Math Track Meet Tests and Answer Keys

University of North Dakota

December 2023

Contents

Grades 7 / 8 Tests and Answer Keys

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 7/8

School ______________

Team Name ______________

Calculators are allowed.

Student Name ______________

1. Find and simplify
$$\frac{5}{6}+\frac{4}{5}-\frac{3}{4}-\frac{2}{3}$$
Write your answer as a simple fraction where the numerator and denominator have no common factors.

(2 pts) 1. ________

2. A class has 22 students. Every student in the class has a dog, a cat, or both a cat and a dog. If 18 students have a dog (with or without a cat), and 9 students have a cat (with or without a dog), how many students have both a cat and a dog?

(3 pts) 2. ________

3. A right triangle has one leg exactly twice as long as the other leg. If the area of the triangle is 25 square centimeters, what is the length of the hypotenuse (in centimeters)? Write your answer to two decimal places.

(3 pts) 3. ________

4. A recipe calls for $1\frac{1}{2}$ cups of sugar and 4 cups of flour. If we increase the recipe to use $2\frac{1}{2}$ cups of sugar, how many cups of flour should we use? Write your answer as a mixed number where the numerator and the denominator of the fraction have no common factors.

(3 pts) 4. ________

5. What is the largest prime factor of 2023?

(3 pts) 5. ________

6. Four students take a quiz. The highest score is 10, and the lowest is 2. If the mean (average) score is 7, what is the median score?

(3 pts) 6. ________

7. A rectangular prism is 8 cm long, 4 cm wide, and 3 cm high. What is the total area (in square cm) of all of the faces of the prism?

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET — INDIVIDUAL TEST #1

University of North Dakota — Grades 7/8

December 19, 2023

School _______________ Team Name _______________

Calculators are allowed. **Key** Student Name _______________

1. Find and simplify

$$\frac{5}{6} + \frac{4}{5} - \frac{3}{4} - \frac{2}{3}$$

Write your answer as a simple fraction where the numerator and denominator have no common factors.

(2 pts) 1. $\frac{13}{60}$

2. A class has 22 students. Every student in the class has a dog, a cat, or both a cat and a dog. If 18 students have a dog (with or without a cat), and 9 students have a cat (with or without a dog), how many students have both a cat and a dog?

(3 pts) 2. 5

3. A right triangle has one leg exactly twice as long as the other leg. If the area of the triangle is 25 square centimeters, what is the length of the hypotenuse (in centimeters)? Write your answer to two decimal places.

(3 pts) 3. 11.18

4. A recipe calls for $1\frac{1}{2}$ cups of sugar and 4 cups of flour. If we increase the recipe to use $2\frac{1}{2}$ cups of sugar, how many cups of flour should we use? Write your answer as a mixed number where the numerator and the denominator of the fraction have no common factors.

(3 pts) 4. $6\frac{2}{3}$

5. What is the largest prime factor of 2023?

(3 pts) 5. 17

6. Four students take a quiz. The highest score is 10, and the lowest is 2. If the mean (average) score is 7, what is the median score?

(3 pts) 6. 8

7. A rectangular prism is 8 cm long, 4 cm wide, and 3 cm high. What is the total area (in square cm) of all of the faces of the prism?

(3 pts) 7. 136

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 7/8

School ______________

Team Name ______________

Calculators are allowed.

Solutions

Student Name ______________

1. Find and simplify

$$\frac{5}{6} + \frac{4}{5} - \frac{3}{4} - \frac{2}{3}$$

Write your answer as a simple fraction where the numerator and denominator have no common factors.

(2 pts) 1. $\frac{13}{60}$

Solution:

$$\frac{5}{6} + \frac{4}{5} - \frac{3}{4} - \frac{2}{3} = \frac{50 + 48 - 45 - 40}{60} = \frac{13}{60}$$

2. A class has 22 students. Every student in the class has a dog, a cat, or both a cat and a dog. If 18 students have a dog (with or without a cat), and 9 students have a cat (with or without a dog), how many students have both a cat and a dog?

(3 pts) 2. 5

Solution: If 22 students have a pet, and 18 have a dog, 4 must only have a cat. If 9 have a cat at all, 5 must have both.

3. A right triangle has one leg exactly twice as long as the other leg. If the area of the triangle is 25 square centimeters, what is the length of the hypotenuse (in centimeters)? Write your answer to two decimal places.

(3 pts) 3. 11.18

Solution: Let the legs have lengths a and $2a$. Then the area must be $\frac{1}{2}a \cdot 2a = a^2$, so that $a = 5$. So the length of the hypotenuse must be $\sqrt{5^2 + 10^2} = \sqrt{125} \approx 11.18$.

4. A recipe calls for $1\frac{1}{2}$ cups of sugar and 4 cups of flour. If we increase the recipe to use $2\frac{1}{2}$ cups of sugar, how many cups of flour should we use? Write your answer as a mixed number where the numerator and the denominator of the fraction have no common factors.

(3 pts) 4. $6\frac{2}{3}$

Solution:

$$\frac{5/2}{3/2} = \frac{x}{4} \Rightarrow x = \frac{20}{3} = 6\frac{2}{3}.$$

5. What is the largest prime factor of 2023?

(3 pts) 5. 17

Solution: $2023 = 7 \cdot 17 \cdot 17$

6. Four students take a quiz. The highest score is 10, and the lowest is 2. If the mean (average) score is 7, what is the median score?

(3 pts) 6. 8

Solution: Since the mean is 7, the total must be $7 \cdot 4 = 28$. Subtracting 10 and 2, the middle two values must sum to 16. They could be 10 and 6, 9 and 7, or 8 and 8, but no matter what, their average, and the median for the whole set, must be 8.

7. A rectangular prism is 8 cm long, 4 cm wide, and 3 cm high. What is the total area (in square cm) of all of the faces of the prism?

(3 pts) 7. 136

Solution: There are 2 faces which are rectangles with sides of lengths equal to each pair of lengths. That is, 2 that are 8 cm by 4 cm, 2 that are 8 cm by 3 cm, and 2 that are 4 cm by 3 cm. So the total surface area is $2 \cdot 8 \cdot 4 + 2 \cdot 8 \cdot 3 + 2 \cdot 4 \cdot 3 = 136\ cm^2$.

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 7/8

School ______________ Team Name ______________

Calculators are **NOT** allowed. Student Name ______________

1. Tom dug a circular quarry in his sandbox that is 8 inches in diameter and 4 inches deep. Callie also dug a circular quarry that is 7 inches in diameter and 5 inches deep. Whose quarry will hold the most water? Tom or Callie?

(2 pts) 1. ________

2. Evaluate the expression: $3 + 3^2 - 3\sqrt{9} + 9 \div 3$

(3 pts) 2. ________

3. Find all solutions for x:

$$x^2 - 4x = -2 - x$$

(3 pts) 3. ________

4. Given values a and b on a real number line, which expression will always give the distance between a and b ?

A. $a - b$ B. $b - a$ C. $|a - b|$ D. $|a + b|$

(3 pts) 4. ________

5. Find the solution to the system of equations and put them in (x, y) form. Simplify answers if possible.

$$-4x + 2y = -5$$
$$8x + y = 5$$

(3 pts) 5. ________

6. Factor completely: $x^4 - 169x^2$

(3 pts) 6. ________

7. Simplify the expression with no negative exponents.

$$\frac{-3a^4b^{-6}c}{18a^{-2}b^7c^5}$$

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 7/8

School _______________ Team Name _______________

Calculators are **NOT** allowed. **Key** Student Name _______________

1. Tom dug a circular quarry in his sandbox that is 8 inches in diameter and 4 inches deep. Callie also dug a circular quarry that is 7 inches in diameter and 5 inches deep. Whose quarry will hold the most water? Tom or Callie?

(2 pts) 1. Tom

2. Evaluate the expression: $3 + 3^2 - 3\sqrt{9} + 9 \div 3$

(3 pts) 2. 6

3. Find all solutions for x:

$$x^2 - 4x = -2 - x$$

(3 pts) 3. $x = 1, 2$

4. Given values a and b on a real number line, which expression will always give the distance between a and b ?

A. $a - b$ B. $b - a$ C. $|a - b|$ D. $|a + b|$

(3 pts) 4. C.

5. Find the solution to the system of equations and put them in (x, y) form. Simplify answers if possible.

$$-4x + 2y = -5$$
$$8x + y = 5$$

(3 pts) 5. $\left(\frac{3}{4}, -1\right)$

6. Factor completely: $x^4 - 169x^2$

(3 pts) 6. $x^2(x + 13)(x - 13)$

7. Simplify the expression with no negative exponents.

$$\frac{-3a^4b^{-6}c}{18a^{-2}b^7c^5}$$

(3 pts) 7. $\frac{-a^6}{6b^{13}c^4}$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 7/8

School ______________

Team Name ______________

Calculators are allowed.

Student Name ______________

1. Evaluate the following expression as a decimal to the nearest thousandth.

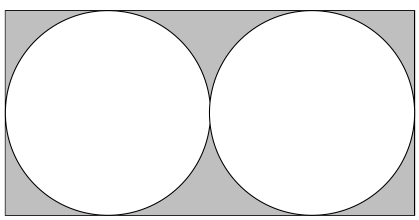
$$\frac{3 + 4 \times 5 + 2^{1+1} + 1}{2^{3+4\div 2}}$$

(2 pts) 1. ________

2. Jack and Jill start running laps of the school track (in the same direction) when the school clock shows the time as 3:00:00 PM. Jack takes 2 minute and 45 seconds to run each lap and Jill takes 3 minutes and 15 seconds. They run until the first time that they reach the starting line at exactly the same time. What time does the school clock show when they stop?

(3 pts) 2. ________

3. The rectangle in the figure below has an area of 32 cm^2. What is the area in cm^2 of the shaded region inside the rectangle and exterior to the circles?



(3 pts) 3. ________

4. Two six-sided dice are rolled. What is the probability that the product of the two numbers is a perfect square?

(3 pts) 4. ________

5. If you add all the dates of the Sundays in a month and the total is 75, what day of the week is the 17th day of the month?

(3 pts) 5. ________

6. Find the set of all x for which $(x + 1)^4 - 5(x + 1)^2 + 4 = 0$.

(3 pts) 6. ________

7. Jay wants to buy a phone which has a regular price of \$449. If Jay lives in a city with 5% sales tax and has saved up \$420, what is the smallest whole number percentage discount on the phone which will allow Jay to buy the phone?

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 7/8

School _______________ Team Name _______________

Calculators are allowed. **Key** Student Name _______________

1. Evaluate the following expression as a decimal to the nearest thousandth.

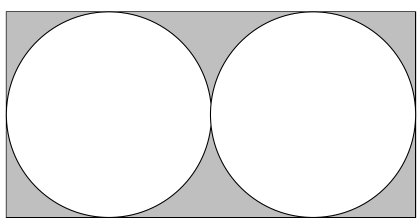
$$\frac{3 + 4 \times 5 + 2^{1+1} + 1}{2^{3+4\div 2}}$$

(2 pts) 1. 0.875

2. Jack and Jill start running laps of the school track (in the same direction) when the school clock shows the time as 3:00:00 PM. Jack takes 2 minute and 45 seconds to run each lap and Jill takes 3 minutes and 15 seconds. They run until the first time that they reach the starting line at exactly the same time. What time does the school clock show when they stop?

(3 pts) 2. 3:35:45 PM

3. The rectangle in the figure below has an area of 32 cm^2. What is the area in cm^2 of the shaded region inside the rectangle and exterior to the circles?



(3 pts) 3. $32 - 8\pi \approx 6.87$

4. Two six-sided dice are rolled. What is the probability that the product of the two numbers is a perfect square?

(3 pts) 4. $\frac{2}{9}$ or 0.22 or 22%

5. If you add all the dates of the Sundays in a month and the total is 75, what day of the week is the 17th day of the month?

(3 pts) 5. Tuesday

6. Find the set of all x for which $(x+1)^4 - 5(x+1)^2 + 4 = 0$.

(3 pts) 6. $\{-3, -2, 0, 1\}$

7. Jay wants to buy a phone which has a regular price of $449. If Jay lives in a city with 5% sales tax and has saved up $420, what is the smallest whole number percentage discount on the phone which will allow Jay to buy the phone?

(3 pts) 7. 11

UND MATHEMATICS TRACK MEET | INDIVIDUAL TEST #3
University of North Dakota | Grades 7/8
December 19, 2023

School ______________ Team Name ______________

Calculators are allowed. **Solutions** Student Name ______________

1. Evaluate the following expression as a decimal to the nearest thousandth.

$$\frac{3+4\times 5+2^{1+1}+1}{2^{3+4\div 2}}$$

(2 pts) 1. 0.875

Solution:

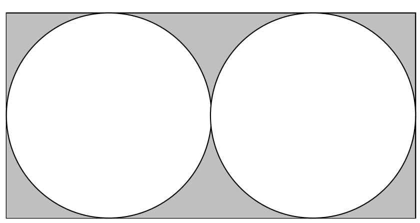
$$\frac{3+4\times 5+2^{1+1}+1}{2^{3+4\div 2}} = \frac{28}{32} = \boxed{0.875}$$

2. Jack and Jill start running laps of the school track (in the same direction) when the school clock shows the time as 3:00:00 PM. Jack takes 2 minute and 45 seconds to run each lap and Jill takes 3 minutes and 15 seconds. They run until the first time that they reach the starting line at exactly the same time. What time does the school clock show when they stop?

(3 pts) 2. 3:35:45 PM

Solution: The LCM of the lap times, 165 seconds and 195 seconds, is 2145 which is 35 minutes 45 seconds. Therefore the time on the clock will be $\boxed{3:35:45\,PM}$ when they reach start line together.

3. The rectangle in the figure below has an area of 32 cm^2. What is the area in cm^2 of the shaded region inside the rectangle and exterior to the circles?



(3 pts) 3. $32-8\pi \approx 6.87$

Solution: The length of rectangle is $4r$ where r is the radius of each circle. The width of the rectangle is $2r$. Therefore $8r^2 = 32$ or $r = 2$. The area inside the rectangle but outside the circles is given by $32 - 2\pi(2)^2 = \boxed{32 - 8\pi = 6.87}$.

4. Two six-sided dice are rolled. What is the probability that the product of the two numbers is a perfect square?

(3 pts) 4. $\frac{2}{9}$ or 0.22 or 22%

Solution: The possibilities for the product of two numbers being a perfect square are $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(4,1)$. Therefore the probability is $\frac{8}{36} = \boxed{\frac{2}{9} = 0.\bar{2}}$ or $\boxed{22\%}$.

5. If you add all the dates of the Sundays in a month and the total is 75, what day of the week is the 17th day of the month?

(3 pts) 5. Tuesday

Solution: The only possibility for the dates a week apart to add up to 75 is if the day of the week lies on the dates $\{1, 8, 15, 22, 29\}$. Therefore the 15th of the month is Sunday which makes the 17th a Tuesday.

6. Find the set of all x for which $(x+1)^4 - 5(x+1)^2 + 4 = 0$.

(3 pts) 6. $\{-3, -2, 0, 1\}$

Solution: Let $u = (x+1)^2$ then $u^2 - 5u + 4 = 0 \Rightarrow (u-4)(u-1) = 0 \Rightarrow u = 1$ or 4. So, $x + 1 = \pm 1$ or $x + 1 = \pm 2$ which gives $x = 0, -2$ or $x = 1, -3$. Therefore the set of solutions is $\boxed{\{-3, -2, 0, 1\}}$

7. Jay wants to buy a phone which has a regular price of \$449. If Jay lives in a city with 5% sales tax and has saved up \$420, what is the smallest whole number percentage discount on the phone which will allow Jay to buy the phone?

(3 pts) 7. 11

Solution: Accounting for tax, the highest price phone Jay can buy is $420/(1.05) = 400$. Therefore, the discount should be at least \$49 which is $49/449 = 10.91\%$ or 11%.

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 7/8

School _______________ Team Name _______________

Calculators are **NOT** allowed. Student Name _______________

1. Evaluate
$$1 + 3|-4^2 - (-8)| + 2|3 + (-5)^2|$$

(2 pts) 1. _________

2. A math club at Oxford High School consists of only ninth-grade and tenth-grade students. If $\frac{3}{10}$ of the students in the math club are from ninth grade and there are 60 more tenth-grade students than ninth-grade students, how many tenth-grade students are there in the math club?

(3 pts) 2. _________

3. Two standard six-sided dice are thrown, and the dice are considered fair. What is the probability that the sum of the two numbers facing up is at least 11?

(3 pts) 3. _________

4. How many gallons of paint would be needed to paint the sides of an uncovered tank that is 100 feet long, 25 feet wide and 15 feet high if one gallon of paint will cover 200 square feet?

(3 pts) 4. _________

5. A factory makes pizza boxes using 8 cutting machines. Each machine cuts 12 boxes every $\frac{3}{4}$ minute. How many boxes can all 8 machines cut in one minute?

(3 pts) 5. _________

6. Find the x-intercept(s) for
$$f(x) = (x-3)^2 - 9$$

(3 pts) 6. _________

7. Simplify the expression
$$-(-3)^2 + 2^3 + 2^{-3} + (-365)^0$$

(3 pts) 7. _________

TOTAL POINTS _________

UND MATHEMATICS TRACK MEET — INDIVIDUAL TEST #4

University of North Dakota — Grades 7/8

December 19, 2023

School ______________ Team Name ______________

Calculators are **NOT** allowed. **Key** Student Name ______________

1. Evaluate
$$1 + 3| - 4^2 - (-8)| + 2|3 + (-5)^2|$$

(2 pts) 1. 81

2. A math club at Oxford High School consists of only ninth-grade and tenth-grade students. If $\frac{3}{10}$ of the students in the math club are from ninth grade and there are 60 more tenth-grade students than ninth-grade students, how many tenth-grade students are there in the math club?

(3 pts) 2. 105

3. Two standard six-sided dice are thrown, and the dice are considered fair. What is the probability that the sum of the two numbers facing up is at least 11?

(3 pts) 3. $\frac{1}{12}$

4. How many gallons of paint would be needed to paint the sides of an uncovered tank that is 100 feet long, 25 feet wide and 15 feet high if one gallon of paint will cover 200 square feet?

(3 pts) 4. 18.75 gallons

5. A factory makes pizza boxes using 8 cutting machines. Each machine cuts 12 boxes every $\frac{3}{4}$ minute. How many boxes can all 8 machines cut in one minute?

(3 pts) 5. 128 boxes

6. Find the x-intercept(s) for
$$f(x) = (x - 3)^2 - 9$$

(3 pts) 6. 0 and 6

7. Simplify the expression
$$-(-3)^2 + 2^3 + 2^{-3} + (-365)^0$$

(3 pts) 7. $\frac{1}{8}$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

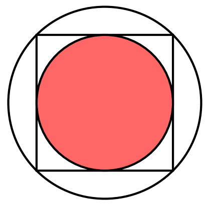
TEAM TEST #1
Grades 7/8

School _______________

Team Name _______________

Calculators are allowed.

1. A square is inscribed in a circle of radius 5. A smaller circle is inscribed in the square. Find the area of the smaller circle.



(20 pts) 1. _________

2. The area of a triangle is 10. If the height of the triangle is twice the base, what is the base?

(20 pts) 2. _________

3. Find positive numbers a and b with $a^2b = 2023$ and $ab^2 = 325$.

(20 pts) 3. _________

4. In a class your grade is determined by the average of five exam scores. If the average of your first three exams is 75%, what must the average of your last two exams be to get an 83% in the class?

(20 pts) 4. _________

5. A bacterial culture doubles every 20 minutes. There are 1500 bacteria at noon. How many bacteria will there be at 4 pm later that same day?

(20 pts) 5. _________

6. What is the greatest common divisor of 360 and 525?

(20 pts) 6. _________

7. Find the positive solution to $2^{x^2-x} = 2 \cdot 8^x$.

(20 pts) 7. _________

8. Suppose the price of an item is increased by 15% and then later the new price is decreased by 10%. Compared to the original price, what percentage increase does the final price represent?

(20 pts) 8. _________

9. Solve the system of equations

$$\begin{cases} 3x + 2y &= 2024 \\ 7x + 5y &= 2023 \end{cases}$$

(20 pts) 9. _________

10. Suppose a and b are positive real numbers such that $a^2 + b^2 = 53$ and $ab = 2$. What is $|a - b|$?

(20 pts) 10. _________

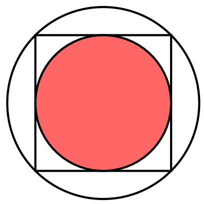
TOTAL POINTS _________

UND MATHEMATICS TRACK MEET | TEAM TEST #1
University of North Dakota | Grades 7/8
December 19, 2023

School ______________ Team Name ______________

Calculators are allowed. **Key**

1. A square is inscribed in a circle of radius 5. A smaller circle is inscribed in the square. Find the area of the smaller circle.



(20 pts) 1. $25\pi/2 \approx 39.27$

2. The area of a triangle is 10. If the height of the triangle is twice the base, what is the base?

(20 pts) 2. $\sqrt{10} \approx 3.16$

3. Find positive numbers a and b with $a^2b = 2023$ and $ab^2 = 325$.

(20 pts) 3. $b \approx 3.74$, $a \approx 23.26$

4. In a class your grade is determined by the average of five exam scores. If the average of your first three exams is 75%, what must the average of your last two exams be to get an 83% in the class?

(20 pts) 4. 95%

5. A bacterial culture doubles every 20 minutes. There are 1500 bacteria at noon. How many bacteria will there be at 4 pm later that same day?

(20 pts) 5. 6144000

6. What is the greatest common divisor of 360 and 525?

(20 pts) 6. 15

7. Find the positive solution to $2^{x^2-x} = 2 \cdot 8^x$.

(20 pts) 7. $2 + \sqrt{5} \approx 4.24$

8. Suppose the price of an item is increased by 15% and then later the new price is decreased by 10%. Compared to the original price, what percentage increase does the final price represent?

(20 pts) 8. 3.5%

9. Solve the system of equations

$$\begin{cases} 3x + 2y = 2024 \\ 7x + 5y = 2023 \end{cases}$$

(20 pts) 9. $x = 6074$, $y = -8099$

10. Suppose a and b are positive real numbers such that $a^2 + b^2 = 53$ and $ab = 2$. What is $|a - b|$?

(20 pts) 10. ____7____

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 7/8

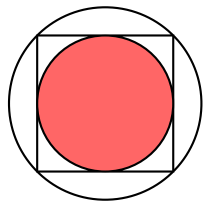
School ______________

Team Name ______________

Calculators are allowed.

Solutions

1. A square is inscribed in a circle of radius 5. A smaller circle is inscribed in the square. Find the area of the smaller circle.



[Sketch of solution: The smaller circle has radius $5\sqrt{2}/2$ by the Pythagorean theorem (or knowledge of 45-45-90 triangles). So the area of the smaller circle is $25\pi/2 \approx 39.27$.]

2. The area of a triangle is 10. If the height of the triangle is twice the base, what is the base?

[Sketch: We have $A = \frac{1}{2}bh$ and $h = 2b$. So solve $10 = \frac{1}{2}b2b$ to find $b = \sqrt{10} \approx 3.16$.]

3. Find positive numbers a and b with $a^2b = 2023$ and $ab^2 = 325$.

[Sketch: Since $a/b = 2023/325$, we have $(2023/325)b^3 = 325$ and so $b \approx 3.74$, $a \approx 23.26$.]

4. In a class your grade is determined by the average of five exam scores. If the average of your first three exams is 75%, what must the average of your last two exams be to get an 83% in the class?

[Sketch: If a, b, c are the first three exam scores and d and e are the last two, we have $(a+b+c)/3 = 75$ and $(a+b+c+d+e)/5 = 83$. Solve to get $d+e = 190$, and so $(d+e)/2 = 95$.]

5. A bacterial culture doubles every 20 minutes. There are 1500 bacteria at noon. How many bacteria will there be at 4 pm later that same day?

[Sketch: The population doubles 12 times between noon and 4 pm so the number of bacteria at 4 pm is $1500 * 2^{12} = 6144000$.]

6. What is the greatest common divisor of 360 and 525?

[Sketch: We have $360 = 2^3 \cdot 3^2 \cdot 5$ and $525 = 3 \cdot 5^2 \cdot 7$ so the GCD is $3 \cdot 5 = 15$.]

7. Find the positive solution to $2^{x^2-x} = 2 \cdot 8^x$.

[Sketch: We have $2^{x^2-x} = 2^{1+3x}$, and so $x^2 - x = 1 + 3x$. The solutions are $2 \pm \sqrt{5}$. The positive solution is $2 + \sqrt{5} \approx 4.24$.]

8. Suppose the price of an item is increased by 15% and then later the new price is decreased by 10%. Compared to the original price, what percentage increase does the final price represent?

   [Sketch: If P is the original price, the final price is $(.9)(1.15)P = (1.035)P$. So the price has increased by 3.5%.]

9. Solve the system of equations

$$\left\{ \begin{array}{rcl} 3x + 2y & = & 2024 \\ 7x + 5y & = & 2023 \end{array} \right.$$

   [Sketch: Use Gaussian elimination or substitution to find $x = 6074$, $y = -8099$.]

10. Suppose a and b are positive real numbers such that $a^2 + b^2 = 53$ and $ab = 2$. What is $|a - b|$?

    [Sketch: Since $(a-b)^2 = a^2 + b^2 - 2ab = 49$ we have $|a - b| = 7$.]

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #2
Grades 7/8

School ______________

Team Name ______________

Calculators are **NOT** allowed.

1. The average weight of five turkeys on a scale is 13 pounds. If a 7-pound turkey is removed from the scale, what is the average weight of the four remaining turkeys?

(20 pts) 1. ________

2. You used $\frac{1}{2}$ can of paint to paint $\frac{3}{5}$ of a wall. How many cans of paint are needed to paint the whole wall?

(20 pts) 2. ________

3. What is the area of a triangle with sides twice as long as those of a triangle with an area of 50 square inches?

(20 pts) 3. ________

4. If S_{66} represents the sum of all the factors of the number 66 and S_{70} represents the sum of all of the factors of the number 70, then find the value of $S_{70} - S_{66}$.

(20 pts) 4. ________

5. The prize money at a raffle was given out as follows:

20 people received \$5
10 people received \$10
5 people received \$20
2 people received \$50
1 person received \$100
1 person received \$500
1 person received \$1000

What is the difference between the mean cash prize and median cash prize?

(20 pts) 5. ________

6. The surface area of a cube is 150 cm^2. What is the volume of the cube?

(20 pts) 6. ________

7. If you use the eight digits $1, 2, 3, 4, 5, 6, 7$, and, 9 each once and only once to form 4 two-digit prime numbers, what will be the sum of the four prime numbers you created?

A. 190 B. 253 C. 172 D. 235 E. None of these

(20 pts) 7. ________

8. If the ratio of the number of widgets X has to the number Z has is two to seven, and their total number of widgets is 1908, how many more widgets does Z have than X?

A. 1050 B. 1060 C. 1484 D. 1030 E. None of these

(20 pts) 8. ________

9. A rectangular solid (shoebox) has dimensions 4 high, 6 wide, and 7 deep. How long is the diagonal through the interior of the solid?

A. $\sqrt{98}$ B. $\sqrt{102}$ C. $\sqrt{85}$ D. $\sqrt{101}$ E. None of these

(20 pts) 9. ________

10. Two cars leave at the same time from the same starting point traveling "down and back." Car A goes 60 mph for 2 hours then returns on the same road going 40 mph. At what constant speed, in mph, would car B need to travel to finish the course at the same time as car A?

(20 pts) 10. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #2
Grades 7/8

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Key

1. The average weight of five turkeys on a scale is 13 pounds. If a 7-pound turkey is removed from the scale, what is the average weight of the four remaining turkeys?

(20 pts) 1. 14.5 pounds

2. You used $\frac{1}{2}$ can of paint to paint $\frac{3}{5}$ of a wall. How many cans of paint are needed to paint the whole wall?

(20 pts) 2. $\frac{5}{6}$

3. What is the area of a triangle with sides twice as long as those of a triangle with an area of 50 square inches?

(20 pts) 3. 200 in^2

4. If S_{66} represents the sum of all the factors of the number 66 and S_{70} represents the sum of all of the factors of the number 70, then find the value of $S_{70} - S_{66}$.

(20 pts) 4. 0

5. The prize money at a raffle was given out as follows:

20 people received \$5
10 people received \$10
5 people received \$20
2 people received \$50
1 person received \$100
1 person received \$500
1 person received \$1000

What is the difference between the mean cash prize and median cash prize?

(20 pts) 5. \$42.50

6. The surface area of a cube is 150 cm^2. What is the volume of the cube?

(20 pts) 6. 125 cm^2

7. If you use the eight digits $1, 2, 3, 4, 5, 6, 7$, and, 9 each once and only once to form 4 two-digit prime numbers, what will be the sum of the four prime numbers you created?

A. 190 B. 253 C. 172 D. 235 E. None of these

(20 pts) 7. A) 190

8. If the ratio of the number of widgets X has to the number Z has is two to seven, and their total number of widgets is 1908, how many more widgets does Z have than X?

A. 1050 B. 1060 C. 1484 D. 1030 E. None of these

(20 pts) 8. B) 1060

9. A rectangular solid (shoebox) has dimensions 4 high, 6 wide, and 7 deep. How long is the diagonal through the interior of the solid?

A. $\sqrt{98}$ B. $\sqrt{102}$ C. $\sqrt{85}$ D. $\sqrt{101}$ E. None of these

(20 pts) 9. D) $\sqrt{101}$

10. Two cars leave at the same time from the same starting point traveling "down and back." Car A goes 60 mph for 2 hours then returns on the same road going 40 mph. At what constant speed, in mph, would car B need to travel to finish the course at the same time as car A?

(20 pts) 10. 48 mph

Grades 9 / 10 Tests and Answer Keys

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 9/10

School ______________

Team Name ______________

Calculators are allowed.

Student Name ______________

1. Each of the numbers 1, 5, 6, 7, 13, 14, 17, 22, and 26 is placed in a different spot below. The numbers 13 and 17 are placed as shown. Joe calculates the average of the numbers in the first three spots, the average of the numbers in the middle three spots, and the average of the numbers in the last three spots. These three averages are equal. What number is placed in the spot to the right of the 17?

___ ___ ___ 13 17 ___ ___ ___ ___

(2 pts) 1. ________

2. A digital clock shows a time of 4:56. How many minutes will pass until the clock next shows a time in which all of the digits are consecutive and are in increasing order?

(3 pts) 2. ________

3. Suppose that p and q are two different prime numbers and that $n = p^2q^2$ What is the number of possible values of n with $n < 1000$?

(3 pts) 3. ________

4. On Monday, Jill traveled x miles at a constant speed of 90 mph. On Tuesday, she traveled on the same route at a constant speed of 120 mph. Her trip on Tuesday took 16 minutes less than her trip on Monday. What is the value of x?

(3 pts) 4. ________

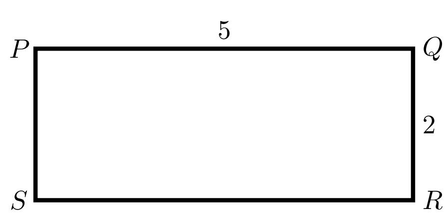
5. A rectangular prism has a volume of 12. A new prism is formed by doubling the length, doubling the width, and tripling the height of the original prism. What is volume of the new prism?

(3 pts) 5. ________

6. In a survey, 100 students were asked if they liked dogs and if they liked cats. A total of 68 students like dogs. A total of 53 like cats. A total of 6 like neither cats nor dogs. How many of the 100 students like both cats and dogs?

(3 pts) 6. ________

7. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point P at an angle of 45° to PQ and bounces off SR. The ball continues to bounce off the sides at 45° until it reaches S. How many bounces of the ball are required?



(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 9/10

School ______________ Team Name ______________

Calculators are allowed. **Key** Student Name ______________

1. Each of the numbers 1, 5, 6, 7, 13, 14, 17, 22, and 26 is placed in a different spot below. The numbers 13 and 17 are placed as shown. Joe calculates the average of the numbers in the first three spots, the average of the numbers in the middle three spots, and the average of the numbers in the last three spots. These three averages are equal. What number is placed in the spot to the right of the 17?

___ ___ ___ 13 17 ___ ___ ___ ___

(2 pts) 1. 7

2. A digital clock shows a time of 4:56. How many minutes will pass until the clock next shows a time in which all of the digits are consecutive and are in increasing order?

(3 pts) 2. 458

3. Suppose that p and q are two different prime numbers and that $n = p^2q^2$ What is the number of possible values of n with $n < 1000$?

(3 pts) 3. 7

4. On Monday, Jill traveled x miles at a constant speed of 90 mph. On Tuesday, she traveled on the same route at a constant speed of 120 mph. Her trip on Tuesday took 16 minutes less than her trip on Monday. What is the value of x?

(3 pts) 4. 96

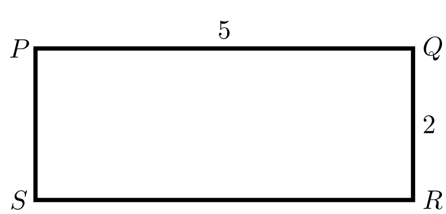
5. A rectangular prism has a volume of 12. A new prism is formed by doubling the length, doubling the width, and tripling the height of the original prism. What is volume of the new prism?

(3 pts) 5. 144

6. In a survey, 100 students were asked if they liked dogs and if they liked cats. A total of 68 students like dogs. A total of 53 like cats. A total of 6 like neither cats nor dogs. How many of the 100 students like both cats and dogs?

(3 pts) 6. 27

7. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point P at an angle of 45° to PQ and bounces off SR. The ball continues to bounce off the sides at 45° until it reaches S. How many bounces of the ball are required?



(3 pts) 7. 5

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 9/10

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Student Name ______________

1. What is the sum of two roots of the equation $3 + \dfrac{1}{x-1} = x - 1$?

(2 pts) 1. ________

2. What is the last digit of 47^{95} ?

(3 pts) 2. ________

3. $x - 1$ divides $2kx^2 + kx + 1$ where k is a constant. What is k ?

(3 pts) 3. ________

4. If $-2 \leq x \leq 3$ and $-4 \leq y \leq 5$, then $a \leq x^2 + y \leq b$. What is $a + b$?

(3 pts) 4. ________

5. If (a, a) is the point on the line $y = x$ closest to the point $(4, 3)$, what is a ?

(3 pts) 5. ________

6. If a sequence $\{a_n\}$ converges to L and $a_{n+1} = \dfrac{2a_n + 3}{4}$, what is L?

(3 pts) 6. ________

7. If $\big||x - 3| - 3\big| \leq 3$, then $a \leq x \leq b$. What is $a + b$?

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET — INDIVIDUAL TEST #2

University of North Dakota — Grades 9/10

December 19, 2023

School ______________ Team Name ______________

Calculators are **NOT** allowed. **Key** Student Name ______________

1. What is the sum of two roots of the equation $3 + \dfrac{1}{x-1} = x - 1$?

   (2 pts) 1. 5

2. What is the last digit of 47^{95} ?

   (3 pts) 2. 3

3. $x - 1$ divides $2kx^2 + kx + 1$ where k is a constant. What is k ?

   (3 pts) 3. $k = -\frac{1}{3}$

4. If $-2 \le x \le 3$ and $-4 \le y \le 5$, then $a \le x^2 + y \le b$. What is $a + b$?

   (3 pts) 4. $a + b = 10$

5. If (a, a) is the point on the line $y = x$ closest to the point $(4, 3)$, what is a ?

   (3 pts) 5. $a = \frac{7}{2}$

6. If a sequence $\{a_n\}$ converges to L and $a_{n+1} = \dfrac{2a_n + 3}{4}$, what is L?

   (3 pts) 6. $L = \frac{3}{2}$

7. If $\big||x - 3| - 3\big| \le 3$, then $a \le x \le b$. What is $a + b$?

   (3 pts) 7. $a + b = 6$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 9/10

School _______________

Team Name _______________

Calculators are **NOT** allowed.

Solutions

Student Name _______________

1. What is the sum of two roots of the equation $3 + \dfrac{1}{x-1} = x - 1$?

(2 pts) 1. 5

Solution: $(x-1)(x-4) = 1 \Rightarrow x^2 - 5x + 3 = 0$

2. What is the last digit of 47^{95} ?

(3 pts) 2. 3

Solution: $47^{95} = (47^4)^{23} \cdot 47^3 \equiv 3 \bmod 10$

3. $x - 1$ divides $2kx^2 + kx + 1$ where k is a constant. What is k ?

(3 pts) 3. $k = -\frac{1}{3}$

Solution: $2kx^2 + kx + 1 = (x-1)(2kx + 3k) + 3k + 1$

4. If $-2 \le x \le 3$ and $-4 \le y \le 5$, then $a \le x^2 + y \le b$. What is $a + b$?

(3 pts) 4. $a+b = 10$

Solution: $0 \le x^2 \le 9 \Rightarrow -4 \le x^2 + y \le 14$

5. If (a, a) is the point on the line $y = x$ closest to the point $(4, 3)$, what is a ?

(3 pts) 5. $a = \frac{7}{2}$

Solution: $(a-4, a-3) \cdot (1, 1) = 0 \Rightarrow 2a - 7 = 0$

6. If a sequence $\{a_n\}$ converges to L and $a_{n+1} = \dfrac{2a_n + 3}{4}$, what is L?

(3 pts) 6. $L = \frac{3}{2}$

Solution: $L = \dfrac{2L+3}{4} \Rightarrow 4L = 2L + 3$

7. If $\big||x-3| - 3\big| \le 3$, then $a \le x \le b$. What is $a + b$?

(3 pts) 7. $a + b = 6$

Solution: $-3 \le |x-3| - 3 \le 3 \Rightarrow |x-3| \le 6 \Rightarrow -6 \le x - 3 \le 6 \Rightarrow -3 \le x \le 9$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 9/10

School ______________

Team Name ______________

Calculators are allowed.

Student Name ______________

1. The y-intercept of the line through $(2, a)$ and $(a, -7)$ with slope $\frac{1}{2}$ is:

(2 pts) 1. ________

2. If $x^3 - x^2 + kx + 3$ is divisible by $x + 1$, then it is also divisible by

   (a) $2x-1$ (b) $2x+2$ (c) x^2-2x+3 (d) x^2+2x+1 (e) x^2-2x+1

(3 pts) 2. ________

3. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

(3 pts) 3. ________

4. The circumference of a circle is 100 inches. The side of a square inscribed in this circle, expressed in inches, is:

   (a) $100\sqrt{2}/\pi$ (b) $50\sqrt{2}$ (c) $100/\pi$ (d) $50\sqrt{2}/\pi$ (e) $25\sqrt{2}/\pi$

(3 pts) 4. ________

5. The number of geese in a flock increases so that the difference between the populations in year $n + 2$ and year n is directly proportional to the population in year $n + 1$. If the populations in the years 2014, 2015, and 2017 were 6, 45, and 117, respectively, then what was the population in 2016?

(3 pts) 5. ________

6. A high school science club plans to rent a minivan for a weekend trip to visit a science museum at a cost of \$120. If they invite two nonmembers along, each member can save \$10 on his or her share of the cost. How many members are in the science club?

(3 pts) 6. ________

7. The graphs of $y = -|x - a| + b$ and $y = |x - c| + d$ intersect at points $(2, 5)$ and $(8, 3)$. Find $a + c$.

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 9/10

School ______________ Team Name ______________

Calculators are allowed. **Key** Student Name ______________

1. The y-intercept of the line through $(2, a)$ and $(a, -7)$ with slope $\frac{1}{2}$ is:

   (2 pts) 1. -5

2. If $x^3 - x^2 + kx + 3$ is divisible by $x + 1$, then it is also divisible by
   (a) $2x-1$ (b) $2x+2$ (c) x^2-2x+3 (d) x^2+2x+1 (e) x^2-2x+1

   (3 pts) 2. (c)

3. An urn is filled with coins and beads, all of which are either silver or gold. Twenty percent of the objects in the urn are beads. Forty percent of the coins in the urn are silver. What percent of the objects in the urn are gold coins?

   (3 pts) 3. 48

4. The circumference of a circle is 100 inches. The side of a square inscribed in this circle, expressed in inches, is:
   (a) $100\sqrt{2}/\pi$ (b) $50\sqrt{2}$ (c) $100/\pi$ (d) $50\sqrt{2}/\pi$ (e) $25\sqrt{2}/\pi$

   (3 pts) 4. (d)

5. The number of geese in a flock increases so that the difference between the populations in year $n + 2$ and year n is directly proportional to the population in year $n + 1$. If the populations in the years 2014, 2015, and 2017 were 6, 45, and 117, respectively, then what was the population in 2016?

   (3 pts) 5. 60

6. A high school science club plans to rent a minivan for a weekend trip to visit a science museum at a cost of \$120. If they invite two nonmembers along, each member can save \$10 on his or her share of the cost. How many members are in the science club?

   (3 pts) 6. 4

7. The graphs of $y = -|x - a| + b$ and $y = |x - c| + d$ intersect at points $(2, 5)$ and $(8, 3)$. Find $a + c$.

   (3 pts) 7. 10

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 9/10

School _______________

Team Name _______________

Calculators are **NOT** allowed.

Student Name _______________

1. Your car is currently 3 years old and is worth \$10,000. Two years ago it was worth \$12,400. Assume the car's value depreciates linearly with time. What will be the value of the car in 5 years?

(2 pts) 1. _________

2. Amtrak's annual passenger revenue for the years 1990-2010 is given by the equation $R = -40|x - 11| + 990$ where R is the annual revenue in millions of dollars and x is the number of years since January 1, 1990. In what years was the passenger revenue \$790 million?

(3 pts) 2. _________

3. Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

(3 pts) 3. _________

4. Students from Mrs. Pohl's class are standing in a circle. They are evenly spaced and consecutively numbered starting with 1. The student with number 3 is standing directly across form the student with number 17. How many students are there in Mrs. Pohl's class?

(3 pts) 4. _________

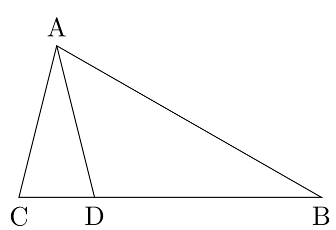
5. Alice and Bill are walking in opposite directions along the same route between A and B. Alice is going from A to B, and Bill from B to A. They start at the same time. They pass each other 3 hours later. Alice arrives at B 2.5 hours before Bill arrives at A. How many hours does it take for Bill to go from B to A?

(3 pts) 5. _________

6. You have two boxes. Each of them has a square base and is half as tall as it is wide. If the larger box is two inches wider than the smaller box, and has a volume 244 in^3 greater, what is the width of the smaller box?

(3 pts) 6. _________

7. In the triangle $\triangle ABC$, the point D lies on side $\overline{BC}$. Also, $AC = 3$, $AD = 3$, $BD = 8$, and $CD = 1$. What is AB?



(3 pts) 7. _________

TOTAL POINTS _________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 9/10

School ______________ Team Name ______________

Calculators are **NOT** allowed. **Key** Student Name ______________

1. Your car is currently 3 years old and is worth \$10,000. Two years ago it was worth \$12,400. Assume the car's value depreciates linearly with time. What will be the value of the car in 5 years?

(2 pts) 1. \$4000

2. Amtrak's annual passenger revenue for the years 1990-2010 is given by the equation $R = -40|x - 11| + 990$ where R is the annual revenue in millions of dollars and x is the number of years since January 1, 1990. In what years was the passenger revenue \$790 million?

(3 pts) 2. 1996 and 2006

3. Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

(3 pts) 3. 60

4. Students from Mrs. Pohl's class are standing in a circle. They are evenly spaced and consecutively numbered starting with 1. The student with number 3 is standing directly across form the student with number 17. How many students are there in Mrs. Pohl's class?

(3 pts) 4. 28

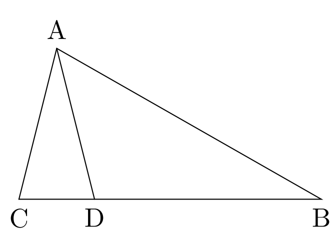
5. Alice and Bill are walking in opposite directions along the same route between A and B. Alice is going from A to B, and Bill from B to A. They start at the same time. They pass each other 3 hours later. Alice arrives at B 2.5 hours before Bill arrives at A. How many hours does it take for Bill to go from B to A?

(3 pts) 5. 7.5 hours

6. You have two boxes. Each of them has a square base and is half as tall as it is wide. If the larger box is two inches wider than the smaller box, and has a volume 244 in^3 greater, what is the width of the smaller box?

(3 pts) 6. 8 in

7. In the triangle $\triangle ABC$, the point D lies on side $\overline{BC}$. Also, $AC = 3$, $AD = 3$, $BD = 8$, and $CD = 1$. What is AB?



(3 pts) 7. 9

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 9/10

School ______________

Team Name ______________

Calculators are allowed.

1. If $x = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ...$, then x is better known as

   a) π b) e c) φ d) none of the previous choices

(20 pts) 1. ________

2. Compute $\left[2^{2^2} \cdot 2 \cdot (2^2 \cdot 2^2)^2 - \begin{pmatrix} 2^2 + 2^{2-2} \\ 2 \end{pmatrix}^2\right]/2^2$.

(20 pts) 2. ________

3. Given a whole number construct a sequence as follows. If the number is even, divide by 2; if it's odd multiply by three and add 1. Apply the same method to the result to get the next number. Stop if/when you reach 1. How many steps are required if you start with 112?

(20 pts) 3. ________

4. In the decimal expansion of 1/14 what is the value of the 30th digit to the right of the decimal point?

(20 pts) 4. ________

5. Find a positive integer m so that

$$1 + 2 + 3 + \cdots + 90 = 100 + 101 + \cdots + m.$$

(20 pts) 5. ________

6. If $f_0 = 0, f_1 = 1$ and for $n \geq 2, f_n = f_{n-1} + f_{n-2}$ find $f_4 + f_9 + f_{16}$.

(20 pts) 6. ________

7. Find a ten digit number which contains every digit 0,1,..., 9 exactly once, starts with a 3 and is divisible by every whole number between 2 and 18.

(20 pts) 7. ________

8. Starting with $x_0 = 1$, for $n \geq 1$ let x_n be the least non-negative remainder when $2 \cdot x_{n-1}$ is divided by $n + 6$. Find the smallest value of n for which $x_n = 0$.

(20 pts) 8. ________

9. Factor 11687 into primes.

(20 pts) 9. ________

10. If $y = \cos\left(\frac{7\pi}{12}\right)$, then the value of y is

    a) $\frac{\sqrt{2} - \sqrt{6}}{4}$ b) $\frac{-\sqrt{2 - \sqrt{3}}}{2}$ c) approximately -0.258819

    d) all of the previous choices

(20 pts) 10. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 9/10

School ______________

Team Name ______________

Calculators are allowed.

Key

1. If $x = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ...$, then x is better known as

   a) π b) e c) φ d) none of the previous choices

(20 pts) 1. b.

2. Compute $\left[2^{2^2} \cdot 2 \cdot (2^2 \cdot 2^2)^2 - \binom{2^2 + 2^{2-2}}{2}^2\right]/2^2$.

(20 pts) 2. 2023

3. Given a whole number construct a sequence as follows. If the number is even, divide by 2; if it's odd multiply by three and add 1. Apply the same method to the result to get the next number. Stop if/when you reach 1. How many steps are required if you start with 112?

(20 pts) 3. 20

4. In the decimal expansion of 1/14 what is the value of the 30th digit to the right of the decimal point?

(20 pts) 4. 8

5. Find a positive integer m so that

$$1 + 2 + 3 + \cdots + 90 = 100 + 101 + \cdots + m.$$

(20 pts) 5. 134

6. If $f_0 = 0, f_1 = 1$ and for $n \geq 2, f_n = f_{n-1} + f_{n-2}$ find $f_4 + f_9 + f_{16}$.

(20 pts) 6. 1024

7. Find a ten digit number which contains every digit 0,1,..., 9 exactly once, starts with a 3 and is divisible by every whole number between 2 and 18.

(20 pts) 7. $3,785,942,160$

8. Starting with $x_0 = 1$, for $n \geq 1$ let x_n be the least non-negative remainder when $2 \cdot x_{n-1}$ is divided by $n + 6$. Find the smallest value of n for which $x_n = 0$.

(20 pts) 8. 18

9. Factor 11687 into primes.

(20 pts) 9. $13 \cdot 29 \cdot 31$

10. If $y = \cos\left(\frac{7\pi}{12}\right)$, then the value of y is

    a) $\frac{\sqrt{2} - \sqrt{6}}{4}$ b) $\frac{-\sqrt{2 - \sqrt{3}}}{2}$ c) approximately -0.258819

    d) all of the previous choices

(20 pts) 10. d.

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 9/10

School ______________

Team Name ______________

Calculators are allowed.

Solutions

1. If $x = 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + ...$, then x is better known as

   a) π b) e c) φ d) none of the previous choices

   The answer is b.

2. Compute $\left[2^{2^2} \cdot 2 \cdot (2^2 \cdot 2^2)^2 - \binom{2^2 + 2^{2-2}}{2}^2\right]/2^2$.

   The answer is 2023 *by straightforward calculation.*

3. Given a whole number construct a sequence as follows. If the number is even, divide by 2; if it's odd multiply by three and add 1. Apply the same method to the result to get the next number. Stop if/when you reach 1. How many steps are required if you start with 112?

   $112 \longrightarrow 56 \longrightarrow 28 \longrightarrow 14 \longrightarrow 7 \longrightarrow 22 \longrightarrow 11 \longrightarrow 34 \longrightarrow 17 \longrightarrow 52 \longrightarrow 26 \longrightarrow 13 \longrightarrow 40 \longrightarrow 20 \longrightarrow 10 \longrightarrow 5 \longrightarrow 16 \longrightarrow 8 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1$. *There are* 20 *arrows.*

4. In the decimal expansion of 1/14 what is the value of the 30th digit to the right of the decimal point?

   The expansion starts 0.0714285714285.... *Since the 6th digit is 8 and the cycle length is 6, the answer is 8.*

5. Find a positive integer m so that $1 + 2 + 3 + \cdots + 90 = 100 + 101 + \cdots + m$.

   The answer is $m = 134$ *which can be seen by using Gauss' formula for* $1+2+3+...+n$ *three times after adding* $1 + 2 + ... + 99$ *to both sides.*

6. If $f_0 = 0, f_1 = 1$ and for $n \geq 2, f_n = f_{n-1} + f_{n-2}$ find $f_4 + f_9 + f_{16}$.

   Using the recursive formula one computes $f_4 + f_9 + f_{16} = 3 + 34 + 987 = 1024$

7. Find a ten digit number which contains every digit 0,1,..., 9 exactly once, starts with a 3 and is divisible by every whole number between 2 and 18.

   If y *represents our number,* y *is divisible by* $2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 12,252,240$. *So* $y = 12,252,240x$, *where* $245 \leq x \leq 326$ *since* $3,000,000,000 \leq y \leq 4,000,000,000$. *Trial and error leads to* $x = 309$ *and* $y = 3,785,942,160$.

8. Starting with $x_0 = 1$, for $n \geq 1$ let x_n be the least non-negative remainder when $2 \cdot x_{n-1}$ is divided by $n + 6$. Find the smallest value of n for which $x_n = 0$.

   One finds the sequence to be $1, 2, 4, 8, 6, 1, 2, 4, 8, 1, 2, 4, 8, 16, 12, 3, 6, 12, 0$ *so* $n = 18$.

9. Factor 11687 into primes.

   The answer is $11687 = 13 \cdot 29 \cdot 31$.

10. If $y = \cos\left(\frac{7\pi}{12}\right)$, then y is

a) $\frac{\sqrt{2}-\sqrt{6}}{4}$ b) $\frac{-\sqrt{2-\sqrt{3}}}{2}$ c) approximately -0.258819 d) all of the previous choices

The answer is d)

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST # 2
Grades 9/10

School _______________ Team Name _______________

Calculators are **NOT** allowed.

1. A cube with edge length 8 feet was molded into a sphere. Find the diameter of the sphere.

(20 pts) 1. _________

2. Find the value of k if the graph of a linear function $y = f(x)$ has slope $m = 3$ and passes through the points $(4, 7)$ and $(k, 1)$.

(20 pts) 2. _________

3. The sum of the squares of two consecutive positive odd integers is 74. What is the value of the larger integer?

(20 pts) 3. _________

4. Find the length of the arc of a circle of radius 10 inches subtended by a central angle of 50°.

(20 pts) 4. _________

5. Suppose the rational function $f(x)$ is given by $f(x) = \dfrac{5x + 7}{2x + 4}$. Find $f^{-1}(x)$.

(20 pts) 5. _________

6. The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes after boarding.

(20 pts) 6. _________

7. If the cost of a bat and a baseball combined is \$1.10 and the ball costs \$1.00 less than the bat, how much is the ball?

(20 pts) 7. _________

8. Let ℓ_1 be the line segment whose endpoints lie at the top left and bottom right corner of a square and let ℓ_2 be the line segment whose endpoints are the bottom left corner of the square and the center of the top side of the square. What fraction of the area of the square is not occupied by the triangle formed from the point at the intersection of ℓ_1 and ℓ_2 and the bottom two corners of the square?

(20 pts) 8. _________

9. The square root of the value obtained by adding twelve to some positive number is the same as the number itself. What is it?

(20 pts) 9. _________

10. Suppose a town has a population of $1,000$ in the year 1999, and suppose the population of the town grows at a constant rate of 125 per year. Find the equation of the line that models the town's population P as a function of t, where t is the number of years since the model began.

(20 pts) 10. _________

TOTAL POINTS _________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST # 2
Grades 9/10

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Key

1. A cube with edge length 8 feet was molded into a sphere. Find the diameter of the sphere.

(20 pts) 1. $8\sqrt[3]{\frac{6}{\pi}}$

2. Find the value of k if the graph of a linear function $y = f(x)$ has slope $m = 3$ and passes through the points $(4, 7)$ and $(k, 1)$.

(20 pts) 2. $k = 2$

3. The sum of the squares of two consecutive positive odd integers is 74. What is the value of the larger integer?

(20 pts) 3. 7

4. Find the length of the arc of a circle of radius 10 inches subtended by a central angle of $50°$.

(20 pts) 4. $\frac{25\pi}{9}$ inches

5. Suppose the rational function $f(x)$ is given by $f(x) = \dfrac{5x + 7}{2x + 4}$. Find $f^{-1}(x)$.

(20 pts) 5. $f^{-1}(x) = \dfrac{7 - 4x}{2x - 5}$

6. The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes after boarding.

(20 pts) 6. $h(t) = -67.5\cos\left(\frac{\pi t}{15}\right) + 69.5$ or $h(t) = -67.5\sin\left(\frac{\pi}{2} - \frac{\pi}{15}t\right) + 69.5$ or $h(t) = 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 69.5$

7. If the cost of a bat and a baseball combined is \$1.10 and the ball costs \$1.00 less than the bat, how much is the ball?

(20 pts) 7. \$0.05

8. Let ℓ_1 be the line segment whose endpoints lie at the top left and bottom right corner of a square and let ℓ_2 be the line segment whose endpoints are the bottom left corner of the square and the center of the top side of the square. What fraction of the area of the square is not occupied by the triangle formed from the point at the intersection of ℓ_1 and ℓ_2 and the bottom two corners of the square?

(20 pts) 8. $\frac{2}{3}$

9. The square root of the value obtained by adding twelve to some positive number is the same as the number itself. What is it?

(20 pts) 9. 4

10. Suppose a town has a population of $1,000$ in the year 1999, and suppose the population of the town grows at a constant rate of 125 per year. Find the equation of the line that models the town's population P as a function of t, where t is the number of years since the model began.

(20 pts) 10. $P = 125t + 1,000$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST # 2
Grades 9/10

School ______________ Team Name ______________

Calculators are **NOT** allowed.

Solutions

1. A cube with edge length 8 feet was molded into a sphere. Find the diameter of the sphere.
   The volume of a cube with sides of length 8 is 8^3. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$, so $8^3 = \frac{4}{3}\pi r^3$, which is to say that $r^3 = \frac{(3)(8^3)}{4\pi}$ and $r = \sqrt[3]{\frac{384}{\pi}}$. Since the diameter D of a sphere is related to the radius by $D = 2r$, we have $D = 2\sqrt[3]{\frac{384}{\pi}} = 8\sqrt[3]{\frac{6}{\pi}}$

2. Find the value of k if the graph of a linear function $y = f(x)$ has slope $m = 3$ and passes through the points $(4, 7)$ and $(k, 1)$.
   Using the point-slope form $y - y_1 = m(x - x_1)$ of a line with slope m passing through a point (x_1, y_1), we can see that our line is given by the function

$$f(x) = 3(x - 4) + 7.$$

So, since a point (a, b) is on the graph of $y = f(x)$ if and only if $f(a) = b$, we may determine the value of k by setting $f(k) = 1$ and solving for k. That is, we obtain our final answer by carrying out the following computation:

$$\begin{aligned} 3(k-4) + 7 &= 1 \\ 3(k-4) &= -6 \\ k - 4 &= -2 \\ k &= 2. \end{aligned}$$

3. The sum of the squares of two consecutive positive odd integers is 74. What is the value of the larger integer?
   Every odd integer is of the form $2k + 1$ for some integer $k \geq 0$. So, the sum of the squares of any two consecutive odd integers is of the form $(2k + 1)^2 + (2k + 3)^2 = 8k^2 + 16k + 10$. If the value of this sum is to be 74, then we need only solve $8k^2 + 16k + 10 = 74$, which amounts to finding the roots of the quadratic $8k^2 + 16k - 64 = 0$, or (factoring out an 8) $k^2 + 2k - 8 = 0$. Via the quadratic formula, this produces $k = -4$ or $k = 2$. Since we are looking for a positive odd integer, we may conclude that $k = 2$. Thus, the smaller integer is $2(2) + 1 = 5$ and the larger is $2(2) + 3 = 7$.

4. Find the length of the arc of a circle of radius 10 inches subtended by a central angle of 50°.
   In a circle of radius r, the length, s, of an arc subtended by an angle with measure θ in radians is $s = r\theta$. So, we first convert 50° to radians:

$$(50^\circ)\left(\frac{\pi \text{ radians}}{180^\circ}\right) = \frac{5\pi}{18} \text{ radians.}$$

Using this, we reach our answer with the following computation:

$$s = (10 \text{ inches})\left(\frac{5\pi}{18}\right) = \left(\frac{25}{9}\right)\pi \text{ inches}$$

5. Suppose the rational function $f(x)$ is given by $f(x) = \frac{5x+7}{2x+4}$. Find $f^{-1}(x)$.
Writing $y = f(x)$, we have $y = \frac{5x+7}{2x+4}$. To find $f^{-1}(x)$, we interchange x and y and solve for y:

$$\begin{aligned} x &= \frac{5y+7}{2y+4} \\ x(2y+4) &= 5y+7 \\ 2xy+4x &= 5y+7 \\ 2xy-5y &= 7-4x \\ (2x-5)y &= 7-4x \\ y &= \frac{7-4x}{2x-5}. \end{aligned}$$

So, replacing y with $f^{-1}(x)$, we have found our inverse function: $f^{-1}(x) = \frac{7-4x}{2x-5}$.

6. The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes after boarding.
With a diameter of 135 meters, the wheel has a radius of 67.5 meters. The height will oscillate with amplitude 67.5 meters above and below the center. That is, $A = 67.5$.

Passengers board 2 meters above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ meters above ground level. Thus, we can conclude that the midline of the oscillation will be at 69.5 meters. That is, $D = 69.5$.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes. Since $P = \frac{2\pi}{B}$, this yields that $30 = \frac{2\pi}{B}$. Solving, this tells us that $B = \frac{2\pi}{30} = \frac{\pi}{15}$.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase.

So, the riders height h (in meters) as function of t (in minutes) is given by any one of the following equivalent answers:

$$\begin{aligned} h(t) &= -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5 \\ &= -67.5\sin\left(\frac{\pi}{2} - \frac{\pi}{15}t\right) + 69.5 \\ &= 67.5\sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 69.5 \end{aligned}$$

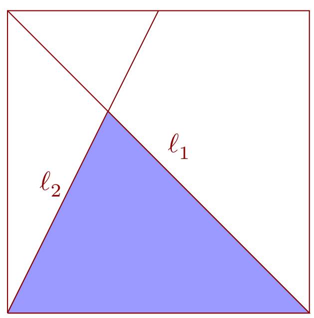
7. If the cost of a bat and a baseball combined is \$1.10 and the ball costs \$1.00 less than the bat, how much is the ball?
Let x denote the cost of the bat and y denote the cost of the ball. We know that $x + y = \$1.10$ and $x = y + \$1.00$. Combining the second equation with

the first, we produce

$$\begin{aligned} x + y &= \$1.10 \\ (y + \$1.00) + y &= \$1.10 \\ 2y + \$1.00 &= \$1.10 \\ 2y &= \$0.10 \\ y &= \$0.05. \end{aligned}$$

Therefore, the ball costs \$0.05.

8. Let ℓ_1 be the line segment whose endpoints lie at the top left and bottom right corner of a square and let ℓ_2 be the line segment whose endpoints are the bottom left corner of the square and the center of the top side of the square. What fraction of the area of the square is not occupied by the triangle formed from the point at the intersection of ℓ_1 and ℓ_2 and the bottom two corners of the square?
Consider a square of side length 1 for simplicity, so that the area of the triangle is the same as the proportion of the square's area that it occupies. The triangle formed is shaded in blue below:



Notice that the height of the triangle and the triangle formed above it sum to 1, and that these two triangles are similar. Let h be the height of the larger triangle, so that $1 - h$ is the height of the smaller. As the ratio of the height to the base of similar triangles is the same, we then have

$$\begin{aligned} \frac{h}{1} &= \frac{1-h}{1/2} \\ h &= 2 - 2h \\ h &= \frac{2}{3}. \end{aligned}$$

So, the area of the triangle is $\frac{1}{2}(1)(h) = \frac{1}{3}$, which is also the proportion of the area of any square a triangle formed in such a fashion will occupy. Hence, the proportion of the square not occupied by the triangle is $\frac{2}{3}$.

9. The square root of the value obtained by adding twelve to some positive number is the same as the number itself. What is it?

Let x denote our sought value. Then, $\sqrt{x+12} = x$, and we have

$$\begin{aligned}
\sqrt{x+12} &= x \\
x + 12 &= x^2 \\
x^2 - x - 12 &= 0 \\
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)} \\
x &= \frac{1 \pm \sqrt{1+48}}{2} \\
x &= \frac{1 \pm 7}{2} \\
x &= 4 \text{ or } -3
\end{aligned}$$

As the number is specified to be positive, we may conclude that it is 4.

10. Suppose a town has a population of $1,000$ in the year 1999, and suppose the population of the town grows at a constant rate of 125 per year. Find the equation of the line that models the town's population P as a function of t, where t is the number of years since the model began.

    Since we are modeling the population of the town with a linear model, the linear function we will produce will be of the form $P = mt + b$ for some real numbers m and b.

    Since our model is linear, the rate of change of the population will be the slope of the line used in our model. That is, since the population of the town grows at a constant rate of 125 per year, $m = 125$.

    Since the variable t corresponds to the number of years since the model began, it must be the case that $1,000 = m(0) + b = b$. So, b must be the initial population of the town, i.e. $b = 1,000$.

    Combining these observations, we reach our final answer and may conclude that our linear model is given by:

$$P = 125t + 1,000.$$

Grades 11 / 12 Tests and Answer Keys

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 11/12

School ______________

Team Name ______________

Calculators are allowed.

Student Name ______________

1. Find the exact value of x satisfying $\log_3 \sqrt{x} = \log_{\sqrt{x}} 3$ and $0 < x < 1$.

(2 pts) 1. ________

2. Given that x and y are distinct nonzero real numbers such that

$$x + \frac{2}{x} = y + \frac{2}{y}$$

what is the value of xy?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2 E. 4

(3 pts) 2. ________

3. What is the square root of the largest perfect square that divides 12! ?

(3 pts) 3. ________

4. A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

A. 1 B. $\frac{\sqrt{6}}{2}$ C. $\sqrt{3}$ D. 2 E. $3\sqrt{2}$

(3 pts) 4. ________

5. The region in the first quadrant bounded by the line $3x+2y=7$ and the coordinate axes is rotated about the x-axis. Approximately, what is the volume of the resulting solid?

A. 8 units^3 B. 20 units^3 C. 30 units^3 D. 90 units^3 E. 120 units^3

(3 pts) 5. ________

6. A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

A. 15 B. 30 C. 40 D. 60 E. 70

(3 pts) 6. ________

7. Let A_1 be the area of a square with a side length of 1. For $n \geq 2$, $A_n = \dfrac{3}{4}A_{n-1}$. Find the value of

$$\frac{A_1}{A_3} \cdot \frac{A_5}{A_7} \cdot \frac{A_9}{A_11} \cdot \ldots \cdot \frac{A_33}{A_35} \cdot \frac{A_35}{A_39}$$

Round your answer to two decimal places.

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 11/12

School _______________

Team Name _______________

Calculators are allowed.

Key

Student Name _______________

1. Find the exact value of x satisfying $\log_3 \sqrt{x} = \log_{\sqrt{x}} 3$ and $0 < x < 1$.

(2 pts) 1. $\frac{1}{9}$

2. Given that x and y are distinct nonzero real numbers such that

$$x + \frac{2}{x} = y + \frac{2}{y}$$

what is the value of xy?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2 E. 4

(3 pts) 2. D 2

3. What is the square root of the largest perfect square that divides 12! ?

(3 pts) 3. $2^5 \cdot 3^2 \cdot 5 =$ 1440

4. A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

A. 1 B. $\frac{\sqrt{6}}{2}$ C. $\sqrt{3}$ D. 2 E. $3\sqrt{2}$

(3 pts) 4. B $\frac{\sqrt{6}}{2}$

5. The region in the first quadrant bounded by the line $3x+2y=7$ and the coordinate axes is rotated about the x-axis. Approximately, what is the volume of the resulting solid?

A. 8 units^3 B. 20 units^3 C. 30 units^3 D. 90 units^3 E. 120 units^3

(3 pts) 5. C 30 units^3

6. A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

A. 15 B. 30 C. 40 D. 60 E. 70

(3 pts) 6. E 70

7. Let A_1 be the area of a square with a side length of 1. For $n \geq 2$, $A_n = \frac{3}{4}A_{n-1}$.
Find the value of

$$\frac{A_1}{A_3} \cdot \frac{A_5}{A_7} \cdot \frac{A_9}{A_11} \cdot \dots \cdot \frac{A_33}{A_35} \cdot \frac{A_35}{A_39}$$

Round your answer to two decimal places.

(3 pts) 7. $\left(\frac{4}{3}\right)^{20} \approx$ 315.34

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #1
Grades 11/12

School ______________

Team Name ______________

Calculators are allowed.

Solutions

Student Name ______________

1. Find the exact value of x satisfying $\log_3 \sqrt{x} = \log_{\sqrt{x}} 3$ and $0 < x < 1$.

(2 pts) 1. $\frac{1}{9}$

Solution $\frac{1}{2}\log_3 x = 2\log_x 3 \Rightarrow (\ln x)^2 = (2\ln 3)^2 \Rightarrow x = 9$ or $x = 1/9$ Since $0 < x < 1$, $x = 1/9$.

2. Given that x and y are distinct nonzero real numbers such that

$$x + \frac{2}{x} = y + \frac{2}{y}$$

what is the value of xy?

A. $\frac{1}{4}$ B. $\frac{1}{2}$ C. 1 D. 2 E. 4

(3 pts) 2. D 2

Solution Multiplying hte given equation by $xy \neq 0$ yields $x^2y + 2y = xy^2 + 2x \Rightarrow xy(x-y) - 2(x-y) = 0 \Rightarrow (xy-2)(x-y) = 0 \Rightarrow xy = 2$.

3. What is the square root of the largest perfect square that divides 12! ?

(3 pts) 3. $2^5 \cdot 3^2 \cdot 5 =$ 1440

Solution $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. The largest perfect square is $2^{10} \cdot 3^4 \cdot 5^2$.

4. A wire is cut into two pieces, one of length a and the other of length b. The piece of length a is bent to form an equilateral triangle, and the piece of length b is bent to form a regular hexagon. The triangle and the hexagon have equal area. What is $\frac{a}{b}$?

A. 1 B. $\frac{\sqrt{6}}{2}$ C. $\sqrt{3}$ D. 2 E. $3\sqrt{2}$

(3 pts) 4. B $\frac{\sqrt{6}}{2}$

Solution The side length of the triangle is $a/3$ and the side length of the hexagon is $b/6$. The hexagon can be subdivided into 6 equilateral triangles by drawing segments from the center of the hexagon to each vertex. The area of the triangle is $\frac{1}{2}(\frac{a}{3})\left(\frac{a}{3}\frac{\sqrt{3}}{2}\right)$. The area of the hexagon is $6 \cdot \frac{1}{2}(\frac{b}{6})\left(\frac{b}{6}\frac{\sqrt{3}}{2}\right)$. Because the areas of the triangle and hexagon are equal, we get $\frac{a^2}{9} = \frac{b^2}{6}$. Thus $\frac{a}{b} = \frac{\sqrt{6}}{2}$.

5. The region in the first quadrant bounded by the line $3x+2y = 7$ and the coordinate axes is rotated about the x-axis. Approximately, what is the volume of the resulting solid?

A. 8 units^3 B. 20 units^3 C. 30 units^3 D. 90 units^3 E. 120 units^3

(3 pts) 5. C 30 units^3

Solution The line $3x + 2y = 7$ has x-intercept 7/3 and y-intercept 7/2. The part of this line that lies in the first quadrant forms a triangle with the coordinate axes. Rotating this triangle about the x-axis produces a cone with radius 7/2 and height 7/3. The volume of this cone is $\frac{1}{3}\pi(\frac{7}{2})^2(\frac{7}{3}) \approx 30$.

6. A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

A. 15 B. 30 C. 40 D. 60 E. 70

(3 pts) 6. E 70

Solution Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore, $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

7. Let A_1 be the area of a square with a side length of 1. For $n \geq 2$, $A_n = \dfrac{3}{4}A_{n-1}$. Find the value of

$$\frac{A_1}{A_3} \cdot \frac{A_5}{A_7} \cdot \frac{A_9}{A_11} \cdot \ldots \cdot \frac{A_33}{A_35} \cdot \frac{A_35}{A_39}$$

Round your answer to two decimal places.

(3 pts) 7. $\left(\dfrac{4}{3}\right)^{20} \approx$ 315.34

Solution $\dfrac{A_1}{A_3} = \dfrac{A_1}{A_2} \cdot \dfrac{A_2}{A_3} = \left(\dfrac{4}{3}\right)^2$. Similarly, each fraction in the expression is $\left(\dfrac{4}{3}\right)^2$, and there are 10 fractions, thus the given expression is $\left(\dfrac{4}{3}\right)^{20} \approx 315.34$

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Student Name ______________

1. Consider two identical beakers, beaker A and beaker B. Beaker B is full of liquid and beaker A is empty. A scientist pours one third of the liquid from beaker B into beaker A. She then pours one fourth of the liquid in beaker A back into beaker B. Finally, she pours half of the liquid in beaker B back into beaker A. After this process, what fraction of the liquid is in beaker A?

   (a) 5/8 (b) 2/3 (c) 5/6 (d) 7/8 (e) none of these

(2 pts) 1. ________

2. Suppose $r, s > 0$. A disk of radius r cm is cut into nine pieces and a second disk of radius $(r + s)$ cm is cut into sixteen pieces. If each of the twenty-five pieces has equal area, then r/s is

   (a) 1/3 (b) 3/4 (c) 4/3 (d) 3 (e) none of these

(3 pts) 2. ________

3. For how many integers n does the equation $x^2+nx+10 = 0$ have integer solutions?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

(3 pts) 3. ________

4. A square $ABCD$ with side 1 is given in the plan. How many points P in the plane of the square satisfy

$$PA + PC + PB + PD = 2\sqrt{2}?$$

   (a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

(3 pts) 4. ________

5. Evan's living room has twice as much area as Victoria's, and three times as much area as Kyle's. Kyle mops floors half as fast as Victoria and one third as fast as Evan. If they all start mopping their floor at the same time, who will finish mopping first?

   (a) Evan (b) Victoria (c) Kyle and Evan tie for first
   (d) Victoria and Kyle tie for first (e) none of these

(3 pts) 5. ________

6. Let $p(x) = 3x^4 + ax^2 + b$ where a, b are real numbers. Suppose that $p(x)$ has exactly three distinct real roots r,s and t with $r < s < t$. Which one of the following statements must be true?

   (a) $r + t = 0$ (b) $s + t = 0$ (c) $rt \geq 0$ (d) $r + s \geq 0$ (e) none of these

(3 pts) 6. ________

7. What is the number of integers m, with $1 \leq m \leq 300$, for which m^m is a perfect cube?

   (a) 100 (b) 101 (c) 103 (d) 104 (e) 106

(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 11/12

School ______________ Team Name ______________

Calculators are **NOT** allowed. **Key** Student Name ______________

1. Consider two identical beakers, beaker A and beaker B. Beaker B is full of liquid and beaker A is empty. A scientist pours one third of the liquid from beaker B into beaker A. She then pours one fourth of the liquid in beaker A back into beaker B. Finally, she pours half of the liquid in beaker B back into beaker A. After this process, what fraction of the liquid is in beaker A?

   (a) 5/8 (b) 2/3 (c) 5/6 (d) 7/8 (e) none of these

   (2 pts) 1. A

2. Suppose $r, s > 0$. A disk of radius r cm is cut into nine pieces and a second disk of radius $(r + s)$ cm is cut into sixteen pieces. If each of the twenty-five pieces has equal area, then r/s is

   (a) 1/3 (b) 3/4 (c) 4/3 (d) 3 (e) none of these

   (3 pts) 2. D

3. For how many integers n does the equation $x^2 + nx + 10 = 0$ have integer solutions?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

   (3 pts) 3. E

4. A square $ABCD$ with side 1 is given in the plan. How many points P in the plane of the square satisfy

   $$PA + PC + PB + PD = 2\sqrt{2}?$$

   (a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

   (3 pts) 4. B

5. Evan's living room has twice as much area as Victoria's, and three times as much area as Kyle's. Kyle mops floors half as fast as Victoria and one third as fast as Evan. If they all start mopping their floor at the same time, who will finish mopping first?

   (a) Evan (b) Victoria (c) Kyle and Evan tie for first
   (d) Victoria and Kyle tie for first (e) none of these

   (3 pts) 5. B

6. Let $p(x) = 3x^4 + ax^2 + b$ where a, b are real numbers. Suppose that $p(x)$ has exactly three distinct real roots r,s and t with $r < s < t$. Which one of the following statements must be true?

   (a) $r + t = 0$ (b) $s + t = 0$ (c) $rt \geq 0$ (d) $r + s \geq 0$ (e) none of these

   (3 pts) 6. A

7. What is the number of integers m, with $1 \leq m \leq 300$, for which m^m is a perfect cube?

   (a) 100 (b) 101 (c) 103 (d) 104 (e) 106

   (3 pts) 7. D

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #2
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Solutions

Student Name ______________

1. Consider two identical beakers, beaker A and beaker B. Beaker B is full of liquid and beaker A is empty. A scientist pours one third of the liquid from beaker B into beaker A. She then pours one fourth of the liquid in beaker A back into beaker B. Finally, she pours half of the liquid in beaker B back into beaker A. After this process, what fraction of the liquid is in beaker A?

   (a) 5/8 (b) 2/3 (c) 5/6 (d) 7/8 (e) none of these

   (2 pts) 1. ___A___

   Solution: If ℓ denotes the amount of liquid in B, then after the first step, the amount of liquid in A is $\frac{1}{3}\ell$ while in B is $\left(1-\frac{1}{3}\right)\ell = \frac{2}{3}\ell$. After the second step, the amount of liquid in A is $\left(\frac{1}{3}-\frac{1}{12}\right)\ell = \frac{1}{4}\ell$ and in B is $\left(\frac{2}{3}+\frac{1}{12}\right)\ell = \frac{3}{4}\ell$. Finally, after the third step, the amount of liquid in A will be $\left(\frac{1}{4}{}_{\frac{3}{8}}\right)\ell = \frac{5}{8}\ell$. The answer is (A).

2. Suppose $r, s > 0$. A disk of radius r cm is cut into nine pieces and a second disk of radius $(r+s)$ cm is cut into sixteen pieces. If each of the twenty-five pieces has equal area, then r/s is

   (a) 1/3 (b) 3/4 (c) 4/3 (d) 3 (e) none of these

   (3 pts) 2. ___D___

   Solution: We have $\frac{\pi r^2}{9} = \frac{\pi(r+s)^2}{16}$ giving $4r = \pm 3(r+s)$. Since $r, s > 0$, it must be that $4r = 3(r+s)$ implying that $r = 3s$. Thus, $r/s = 3$. The answer is (D).

3. For how many integers n does the equation $x^2 + nx + 10 = 0$ have integer solutions?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

   (3 pts) 3. ___E___

   Solution: The product of the solutions must be 10, which means that the solutions must be
   $\{\pm(1,10), \pm(2,5)\}$. This leads to $n = \pm 11, \pm 7$. The answer is (E).

4. A square $ABCD$ with side 1 is given in the plan. How many points P in the plane of the square satisfy

   $$PA + PC + PB + PD = 2\sqrt{2}?$$

   (a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

   (3 pts) 4. ___B___

   Solution: We have $PA + PC \geq AC$, $PB + PD \geq BD$ and the equality in both inequalities is obtained when P is on AC and BD. i.e.,P is the intersection of the diagonals of the square. Therefore.

   $$2\sqrt{2} = PA + PC + PB + PD \geq AC + BD = 2\sqrt{2}$$

   Hence, $PA + PC + PB + PD = AC + BD$. Thus, P is the center of the square. The answer is (B).

5. Evan's living room has twice as much area as Victoria's, and three times as much area as Kyle's. Kyle mops floors half as fast as Victoria and one third as fast as Evan. If they all start mopping their floor at the same time, who will finish mopping first?

(a) Evan (b) Victoria (c) Kyle and Evan tie for first
(d) Victoria and Kyle tie for first (e) none of these

(3 pts) 5. B

Solution: If a denotes Evan's living room area, then Victoria's living room area is $\frac{a}{2}$ and Kyle's living room area is $\frac{a}{3}$. If v denotes Evan's mopping speed, then Kyle's mopping speed is $\frac{v}{3}$ and Victoria's mopping speed is $\frac{2v}{3}$. We conclude that Evan mops the floor in $\frac{a}{v}$ units of time, which is equal to Kyle's time, while Victoria mops the floor in $\frac{3a}{4v}$ units of time. Therefore, Victoria will be the first person who finishes mopping. Therefore, the answer is (B).

6. Let $p(x) = 3x^4 + ax^2 + b$ where a, b are real numbers. Suppose that $p(x)$ has exactly three distinct real roots r,s and t with $r < s < t$. Which one of the following statements must be true?

(a) $r + t = 0$ (b) $s + t = 0$ (c) $rt \geq 0$ (d) $r + s \geq 0$ (e) none of these

(3 pts) 6. A

Solution: Observe that if k is a root of $p(x)$ then $p(-k) = p(k)$, implying that $-k$ is also a root of $p(x)$. Thus we have $r = -t$ and $s = 0$ and so $r < 0 < t$. This implies that $s + t > 0$, $rt < 0$, $r + s < 0$ and $r + t = 0$. The answer is (A).

7. What is the number of integers m, with $1 \leq m \leq 300$, for which m^m is a perfect cube?

(a) 100 (b) 101 (c) 103 (d) 104 (e) 106

(3 pts) 7. D

Solution: If $m = 3k$ then $m^m = \left(m^k\right)^3$, hence m^m is a perfect cube. If $m = 3k+1$, then $m^m = \left(m^k\right)^3 m$ which is a perfect cube if m is a perfect cube. If $m = 3k + 2$, then $m^m = \left(m^k\right)^3 m^2$, which is a perfect cube if m is a perfect cube. Between 1 and 300 there are 100 multiples of 3, and four numbers of the form $3k + 1$ or $3k + 2$ which are perfect cubes $(1, 8 = 2^3, 64 = 4^3$ and $125 = 5^3)$. Therefore there are 104 integers m having the desired properties. The answer is (D).

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 11/12

School ______________ Team Name ______________

Calculators are allowed. Student Name ______________

1. In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

(2 pts) 1. _________

2. Suppose there are many socks of 4 different colors in a box: red, black, blue and yellow. Socks are randomly picked from the box one by one. What is the minimum number of socks that need to be picked from the box before 3 pairs of socks can be guaranteed? The pairs do not need to match each other, but socks within a pair must match.

(3 pts) 2. _________

3. Two different positive numbers x and y each differ from their reciprocals by 1. What is the sum of x and y ?

(3 pts) 3. _________

4. David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

(3 pts) 4. _________

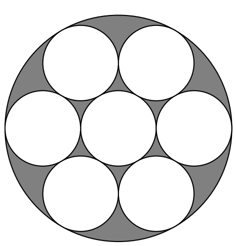
5. Compute the sum of all roots of $(4x + 5)(x - 6) + (x - 3)(4x + 5) = 0$

(3 pts) 5. _________

6. What is the largest prime number that divides 2023 ?

(3 pts) 6. _________

7. The largest circle in the figure has radius one. Seven circles are arranged in the largest circle such that the innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region. Use $\pi = 3.14$ and round your answer to three decimal places.



(3 pts) 7. _________

TOTAL POINTS _________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #3
Grades 11/12

School ______________ Team Name ______________

Calculators are allowed. **Key** Student Name ______________

1. In a baseball conference with 8 teams, how many games must be played so that each team plays every other team exactly once?

(3 pts) 1. 28

2. Suppose there are many socks of 4 different colors in a box: red, black, blue and yellow. Socks are randomly picked from the box one by one. What is the minimum number of socks that need to be picked from the box before 3 pairs of socks can be guaranteed? The pairs do not need to match each other, but socks within a pair must match.

(2 pts) 2. 9

3. Two different positive numbers x and y each differ from their reciprocals by 1. What is the sum of x and y ?

(3 pts) 3. $\sqrt{5}$ or 2.236

4. David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

(3 pts) 4. 210

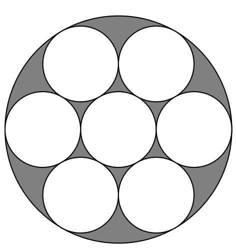
5. Compute the sum of all roots of $(4x + 5)(x - 6) + (x - 3)(4x + 5) = 0$

(3 pts) 5. $\frac{13}{4}$ or 3.25

6. What is the largest prime number that divides 2023 ?

(3 pts) 6. 17

7. The largest circle in the figure has radius one. Seven circles are arranged in the largest circle such that the innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region. Use $\pi = 3.14$ and round your answer to three decimal places.



(3 pts) 7. 0.698

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 11/12

School ______________ Team Name ______________

Calculators are **NOT** allowed. Student Name ______________

1. Let $x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. Which of the following is true?

   a. $x < 2$ b. $x = 2$ c. $x > 2$

(2 pts) 1. ________

2. If $(x-1)$ is a factor of the polynomial $P(x) = x^7 - 4x^4 + 2x^3 + 3x^2 + ax - 8$, find a.

(3 pts) 2. ________

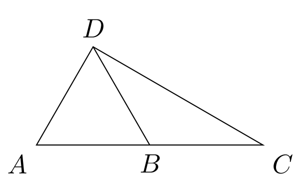
3. Suppose a and b are integers with $2a - 3b = 2$. Find $\dfrac{9^a}{27^b}$.

(3 pts) 3. ________

4. Sally and Bob play the following game: Four fair coins are flipped. If exactly one of them comes up heads Bob wins. If exactly 1 of them comes up tails Bob wins. In all other cases Sally wins. What is the probability that Sally wins?

(3 pts) 4. ________

5. In the picture below assume that A, B, and C are collinear, $\triangle ABD$ is equilateral, and that AB= BC = 1. What is CD?

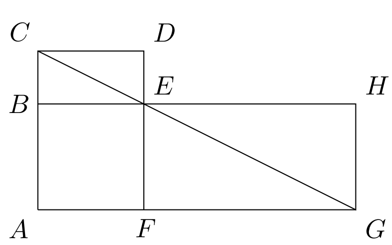


(3 pts) 5. ________

6. Find x if x is positive and $\dfrac{1}{\frac{1}{x+1}+\frac{1}{x+4}} = x$.

(3 pts) 6. ________

7. In the picture below $\square AGHB$ and $\square ACDF$ are rectangles and C, G, and E are collinear. If $AF = AB = 2$ and $BC = 1$, what is EG?



(3 pts) 7. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 11/12

School _______________ Team Name _______________

Calculators are **NOT** allowed. **Key** Student Name _______________

1. Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}$. Which of the following is true?

   a. $x < 2$ b. $x = 2$ c. $x > 2$

(2 pts) 1. a

2. If $(x - 1)$ is a factor of the polynomial $P(x) = x^7 - 4x^4 + 2x^3 + 3x^2 + ax - 8$, find a.

(3 pts) 2. 6

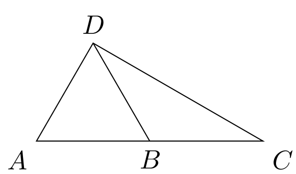
3. Suppose a and b are integers with $2a - 3b = 2$. Find $\frac{9^a}{27^b}$.

(3 pts) 3. 9

4. Sally and Bob play the following game: Four fair coins are flipped. If exactly one of them comes up heads Bob wins. If exactly 1 of them comes up tails Bob wins. In all other cases Sally wins. What is the probability that Sally wins?

(3 pts) 4. 50% or 1/2

5. In the picture below assume that A, B, and C are collinear, $\triangle ABD$ is equilateral, and that AB= BC = 1. What is CD?

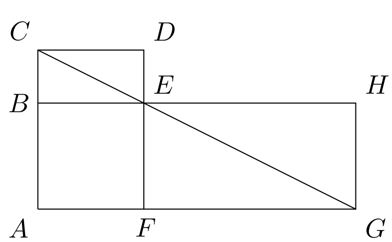


(3 pts) 5. $\sqrt{3}$

6. Find x if x is positive and $\dfrac{1}{\frac{1}{x+1} + \frac{1}{x+4}} = x$.

(3 pts) 6. 2

7. In the picture below $\square AGHB$ and $\square ACDF$ are rectangles and C, G, and E are collinear. If $AF = AB = 2$ and $BC = 1$, what is EG?



(3 pts) 7. $2\sqrt{5}$

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University of North Dakota
December 19, 2023

INDIVIDUAL TEST #4
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Solutions

Student Name ______________

1. Let $x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$. Which of the following is true?

   a. $x < 2$ b. $x = 2$ c. $x > 2$

(2 pts) 1. a

Since $\sqrt{2} < 2$, $2+\sqrt{2} < 4$ and $\sqrt{2+\sqrt{2}} < 2$. We can repeat this kind of reasoning 3 more times to find $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} < 2$.

2. If $(x-1)$ is a factor of the polynomial $P(x) = x^7 - 4x^4 + 2x^3 + 3x^2 + ax - 8$, find a.

(3 pts) 2. 6

If $x-1$ is a factor of $P(x)$, then $P(1) = 0$. So:

$$P(1) = 1 - 4 + 2 + 3 + a - 8 = 0$$
$$a = 6$$

3. Suppose a and b are integers with $2a - 3b = 2$. Find $\frac{9^a}{27^b}$.

(3 pts) 3. 9

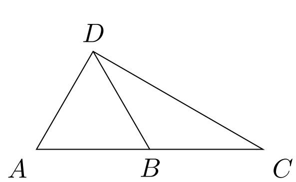
$$\frac{9^a}{27^b} = \frac{3^{2a}}{3^{3b}} = 3^{2a-3b} = 3^2 = 9$$

4. Sally and Bob play the following game: Four fair coins are flipped. If exactly one of them comes up heads Bob wins. If exactly 1 of them comes up tails Bob wins. In all other cases Sally wins. What is the probability that Sally wins?

(3 pts) 4. 50% or 1/2

There are $2^4 = 16$ outcomes when flipping 4 coins. Of these there are 4 outcomes with exactly 1 head and 4 with exactly 1 tail. The other 8 outcomes result in a win for Sally, so her probability of winning is 8/16 = 1/2. Note: also accept 50% or 0.5, etc.

5. In the picture below assume that A, B, and C are collinear, $\triangle ABD$ is equilateral, and that AB= BC = 1. What is CD?



(3 pts) 5. $\sqrt{3}$

The conditions imply that angle $\angle ADC$ is a right angle since it is inscribed in a semicircle. Now $AD = 1$ and $AC = 2$, so $CD = \sqrt{2^2 - 1^2} = \sqrt{3}$ by the Pythagorean Theorem.

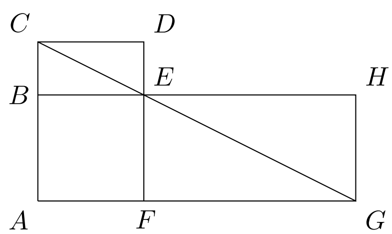
6. Find x if x is positive and $\dfrac{1}{\frac{1}{x+1} + \frac{1}{x+4}} = x$.

(3 pts) 6. 2

$$\begin{aligned} \frac{1}{\frac{1}{x+1} + \frac{1}{x+4}} &= x && \textit{multiply the left side by } \frac{(x+1)(x+4)}{(x+1)(x+4)} \\ \frac{(x+1)(x+4)}{(x+4)+(x+1)} &= x \\ \frac{x^2+5x+4}{2x+5} &= x \\ x^2+5x+4 &= 2x^2+5x \\ x^2 &= 4 \end{aligned}$$

Since $x > 0$ we must have $x = 2$.

7. In the picture below $\square AGHB$ and $\square ACDF$ are rectangles and C, G, and E are collinear. If $AF = AB = 2$ and $BC = 1$, what is EG?



(3 pts) 7. $2\sqrt{5}$

Note that $\triangle BCE$ and $\triangle FEG$ are similar. Since $EF = 2BC$, it follows that $EG = 2CE$. The Pythagorean Theorem implies that $CE = \sqrt{1^2 + 2^2} = \sqrt{5}$, so $EG = 2\sqrt{5}$.

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 11/12

School ______________ Team Name ______________

Calculators are allowed.

1. A rectangular garden is 75 feet long and 32 feet wide. Find the area of the garden. Express your answer in square feet.

(20 pts) 1. ________

2. The wheels of a car are exactly 2 feet in diameter. As the car travels, the wheels of the car rotate at the rate of 8 revolutions per second. How fast is the car traveling? Express your answer in feet per second, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 2. ________

3. The volume of a certain sphere is exactly 75 cubic inches. Find the radius of the sphere. Express your answer in inches, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 3. ________

4. A deck of 52 cards has exactly 12 "face cards," i.e. cards with a king, queen, or jack. Suppose you randomly select three cards from the deck. Find the probability that none of the selected cards is a face card. Round your answer to three decimal places.

(20 pts) 4. ________

5. Mrs. Lopez plans to invest a total of \$50,000 in two different types of bonds: bonds of type A and bonds of type B. Bonds of type A return a dividend of 4% per year, and bonds of type B return a dividend of 7% per year. For example, if Mrs. Lopez invests \$40,000 in bonds of type A and \$10,000 in bonds of type B, then the bonds will return a total dividend of $\$40{,}000 \times 0.04 + \$10{,}000 \times 0.07 = \$2{,}300$ per year. How much money should Mrs. Lopez invest in each type of bond if the bonds are to return a total dividend of \$3,000 per year? Round your answers to the nearest cent.

(20 pts) 5. ________

6. Mrs. Redfeather is designing a box. The top, the bottom, and the sides of the box are to be rectangles, but these rectangles are not necessarily of the same size. The dimensions of the box are to be x inches by $10 - 2x$ inches by $7 - 2x$ inches. Here x is a number which Mrs. Redfeather has not yet determined. Find the value of x which will result in the box with the largest possible volume. Round your answer to two decimal places. Hint: All three dimensions must be positive real numbers.

(20 pts) 6. ________

7. Let distinct real numbers a, b, and c be given. Suppose that a certain polynomial $p(x)$ has real coefficients, and suppose that when $p(x)$ is divided by $x-a$, $x-b$, and $x-c$, it leaves remainders a, b, and c, respectively. What is the remainder when $p(x)$ is divided by $(x-a)(x-b)(x-c)$?

(20 pts) 7. ________

8. What is the smallest positive integer n for which 136^n will have more than 1000 digits?

(20 pts) 8. ________

9. Suppose that $p(x)$ is a polynomial of degree 2 such that $p(1) = 1$, $p(2) = \frac{1}{2}$, and $p(3) = \frac{1}{3}$. Find $p(4)$.

(20 pts) 9. ________

10. Consider a group of 6 people. Each of the six people knows exactly one piece of information, and all 6 pieces of information are different. Every time person "A" calls person "B" on the telephone, "A" tells "B" everything he or she knows, while "B" tells "A" nothing. What is the minimum number of telephone calls between pairs of people needed for everyone to know everything?

(20 pts) 10. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #1
Grades 11/12

School ______________

Team Name ______________

Calculators are allowed.

Key

1. A rectangular garden is 75 feet long and 32 feet wide. Find the area of the garden. Express your answer in square feet.

(20 pts) 1. 2400 sq ft

2. The wheels of a car are exactly 2 feet in diameter. As the car travels, the wheels of the car rotate at the rate of 8 revolutions per second. How fast is the car traveling? Express your answer in feet per second, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 2. 50.27 ft/sec

3. The volume of a certain sphere is exactly 75 cubic inches. Find the radius of the sphere. Express your answer in inches, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 3. 2.62 in

4. A deck of 52 cards has exactly 12 "face cards," i.e. cards with a king, queen, or jack. Suppose you randomly select three cards from the deck. Find the probability that none of the selected cards is a face card. Round your answer to three decimal places.

(20 pts) 4. 0.447

5. Mrs. Lopez plans to invest a total of \$50,000 in two different types of bonds: bonds of type A and bonds of type B. Bonds of type A return a dividend of 4% per year, and bonds of type B return a dividend of 7% per year. For example, if Mrs. Lopez invests \$40,000 in bonds of type A and \$10,000 in bonds of type B, then the bonds will return a total dividend of \$40,000 $\times$ 0.04 + \$10,000 $\times$ 0.07 = \$2,300 per year. How much money should Mrs. Lopez invest in each type of bond if the bonds are to return a total dividend of \$3,000 per year? Round your answers to the nearest cent.

(20 pts) 5. A: \$16,666.67, B: \$33,333.33

6. Mrs. Redfeather is designing a box. The top, the bottom, and the sides of the box are to be rectangles, but these rectangles are not necessarily of the same size. The dimensions of the box are to be x inches by $10 - 2x$ inches by $7 - 2x$ inches. Here x is a number which Mrs. Redfeather has not yet determined. Find the value of x which will result in the box with the largest possible volume. Round your answer to two decimal places. Hint: All three dimensions must be positive real numbers.

(20 pts) 6. 1.35 in

7. Let distinct real numbers a, b, and c be given. Suppose that a certain polynomial $p(x)$ has real coefficients, and suppose that when $p(x)$ is divided by $x-a$, $x-b$, and $x-c$, it leaves remainders a, b, and c, respectively. What is the remainder when $p(x)$ is divided by $(x-a)(x-b)(x-c)$?

(20 pts) 7. $\underline{\quad x \quad}$

8. What is the smallest positive integer n for which 136^n will have more than 1000 digits?

(20 pts) 8. $\underline{\ n = 469\ }$

9. Suppose that $p(x)$ is a polynomial of degree 2 such that $p(1) = 1$, $p(2) = \frac{1}{2}$, and $p(3) = \frac{1}{3}$. Find $p(4)$.

(20 pts) 9. $\underline{\quad \frac{1}{2} \quad}$

10. Consider a group of 6 people. Each of the six people knows exactly one piece of information, and all 6 pieces of information are different. Every time person "A" calls person "B" on the telephone, "A" tells "B" everything he or she knows, while "B" tells "A" nothing. What is the minimum number of telephone calls between pairs of people needed for everyone to know everything?

(20 pts) 10. 10 calls

UND MATHEMATICS TRACK MEET — TEAM TEST #1
University of North Dakota — Grades 11/12
December 19, 2023

School ______________ Team Name ______________

Calculators are allowed.

Solutions

1. A rectangular garden is 75 feet long and 32 feet wide. Find the area of the garden. Express your answer in square feet.

(20 pts) 1. 2400 sq ft

2. The wheels of a car are exactly 2 feet in diameter. As the car travels, the wheels of the car rotate at the rate of 8 revolutions per second. How fast is the car traveling? Express your answer in feet per second, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 2. 50.27 ft/sec

Solution: The circumference is $2\pi r = 2\pi$ feet. The speed is

$$\frac{8 \text{ rev}}{\text{sec}} \times \frac{2\pi \text{ ft}}{\text{rev}} = \frac{16\pi \text{ ft}}{\text{sec}} \approx 50.27 \frac{\text{ft}}{\text{sec}}$$

3. The volume of a certain sphere is exactly 75 cubic inches. Find the radius of the sphere. Express your answer in inches, and round your answer to two decimal places. You may assume that $\pi = 3.14159$.

(20 pts) 3. 2.62 in

Solution: If r is the radius, then the volume is $V = \frac{4}{3}\pi r^3$. Solving, this means

$$r = \sqrt[3]{\frac{3(75)}{4\pi}} \approx 2.62 \text{ in}$$

4. A deck of 52 cards has exactly 12 "face cards," i.e. cards with a king, queen, or jack. Suppose you randomly select three cards from the deck. Find the probability that none of the selected cards is a face card. Round your answer to three decimal places.

(20 pts) 4. 0.447

Solution: Imagine that you select the cards in succession. So you randomly select one card. Then you select another card from the remaining 51 cards. Then you select a third card from the remaining 50 cards. We let $P(\text{event})$ denote the probability of the event.

$$\begin{aligned}
P(\text{first card is not a face card}) &= \frac{40}{52} \\
P(\text{second card is not a face card, given that the first wasn't}) &= \frac{39}{51} \\
P(\text{third card is not, given that the first two weren't}) &= \frac{38}{50} \\
P(\text{none are face cards}) &= \frac{40}{52} \cdot \frac{39}{51} \cdot \frac{38}{50} \approx 0.447
\end{aligned}$$

5. Mrs. Lopez plans to invest a total of \$50,000 in two different types of bonds: bonds of type A and bonds of type B. Bonds of type A return a dividend of 4% per year, and bonds of type B return a dividend of 7% per year. For example, if Mrs. Lopez invests \$40,000 in bonds of type A and \$10,000 in bonds of type B, then the bonds will return a total dividend of \$40,000 × 0.04 + \$10,000 × 0.07 = \$2,300 per year. How much money should Mrs. Lopez invest in each type of bond if the bonds are to return a total dividend of \$3,000 per year? Round your answers to the nearest cent.

(20 pts) 5. A: \$16,666.67, B: \$33,333.33

Solution: Let A be the amount of money, in dollars, that Mrs. Lopez invests in bonds of type A. Let B be the amount of money, in dollars, that Mrs. Lopez invests in bonds of type B. Then

$$\begin{cases} A + B = 50,000 \\ 0.04A + 0.07B = 3000 \end{cases}$$

But then $B = 50,000 - A$, so

$$\begin{aligned} 0.04A + 0.07(50,000 - A) &= 3000 \\ 0.07(50,000) + 0.04A - 0.07A &= 3000 \\ 0.07(50,000) - 3000 &= 0.03A \\ \frac{0.07(50,000) - 3000}{0.03} &= A \approx 16,666.67, \text{ and} \\ B &= 50,000 - A \approx 33,333.33 \end{aligned}$$

6. Mrs. Redfeather is designing a box. The top, the bottom, and the sides of the box are to be rectangles, but these rectangles are not necessarily of the same size. The dimensions of the box are to be x inches by $10 - 2x$ inches by $7 - 2x$ inches. Here x is a number which Mrs. Redfeather has not yet determined. Find the value of x which will result in the box with the largest possible volume. Round your answer to two decimal places. Hint: All three dimensions must be positive real numbers.

(20 pts) 6. 1.35 in

Solution: All three dimensions must be positive real numbers. So $x > 0$, $10-2x > 0$, and $7 - 2x > 0$, which means that $5 > x$ and $\frac{7}{2} > x$. Thus $0 < x < \frac{7}{2} = 3.5$. The volume of the box will be $V(x) = x(10 - 2x)(7 - 2x)$. Graphing this function on a calculator, we can zoom in to see that the largest V occurs when $x \approx 1.35$ in.

7. Let distinct real numbers a, b, and c be given. Suppose that a certain polynomial $p(x)$ has real coefficients, and suppose that when $p(x)$ is divided by $x-a$, $x-b$, and $x - c$, it leaves remainders a, b, and c, respectively. What is the remainder when $p(x)$ is divided by $(x - a)(x - b)(x - c)$?

(20 pts) 7. x

Solution: By the division algorithm, there are polynomials $q_a(x)$, $q_b(x)$, and $q_c(x)$ such that

$$\begin{aligned} p(x) &= (x - a)q_a(x) + a \\ p(x) &= (x - b)q_b(x) + b \\ p(x) &= (x - c)q_c(x) + c. \end{aligned}$$

Thus,

$$p(a) = a, \qquad p(b) = b, \qquad \text{and} \qquad p(c) = c. \tag{*}$$

We may apply the division algorithm again to find that

$$p(x) = (x-a)(x-b)(x-c)q(x) + r(x). \tag{**}$$

Here $q(x)$ and $r(x)$ are polynomials, and $r(x)$ has degree at most 2. By (*) and (**),

$$r(a) = p(a) = a, \quad r(b) = p(b) = b, \quad \text{and} \quad r(c) = p(c) = c.$$

Thus, $r(a) = a$, $r(b) = b$, and $r(c) = c$.

Now let $s(x) = r(x) - x$. Then $s(x)$ has degree at most 2. But $s(a) = r(a) - a = a - a = 0$, $s(b) = r(b) - b = b - b = 0$, and $s(c) = r(c) - c = c - c = 0$, so that $s(x)$ has three distinct zeros. Thus, $s(x) = 0$ for all real x, and $r(x) = x$ for all real x. Therefore, the desired remainder is the polynomial x.

8. What is the smallest positive integer n for which 136^n will have more than 1000 digits?

(20 pts) 8. $n = 469$

Solution: If we were to write out the number 10^{1000}, we would write the digit 1 followed by 1000 zeros. Thus 10^{1000} is the smallest positive integer with more than 1000 digits. We see

$$\begin{aligned} 136^x &= 10^{1000} \qquad (*) \\ \log_{10} 136^x &= \log_{10} 10^{1000} \\ x \log_{10} 136 &= 1000 \\ x &= \frac{1000}{\log_{10} 136} \approx 468.7. \end{aligned}$$

Thus, $136^{468} < 10^{1000} < 136^{469}$, so that 136^{468} has fewer than 1001 digits, and 136^{469} has at least 1001 digits. So the smallest integer n for which 136^n has more than 1000 digits is $n = 469$.

9. Suppose that $p(x)$ is a polynomial of degree 2 such that $p(1) = 1$, $p(2) = \frac{1}{2}$, and $p(3) = \frac{1}{3}$. Find $p(4)$.

(20 pts) 9. $\frac{1}{2}$

Solution: Consider the polynomial $q(x) = xp(x) - 1$, which has degree 3. Also note that

$$\begin{aligned} q(1) &= 1p(1) - 1 = 1 \cdot 1 - 1 = 0 \\ q(2) &= 2p(2) - 1 = 2 \cdot \frac{1}{2} - 1 = 0 \\ q(3) &= 3p(3) - 1 = 3 \cdot \frac{1}{3} - 1 = 0, \end{aligned}$$

so that $q(x) = c(x-1)(x-2)(x-3)$ for some nonzero constant c. But $q(0) = 0p(0) - 1 = -1 = c(-1)(-2)(-3) = -6c$, so that $c = \frac{1}{6}$. Therefore,

$$p(4) = \frac{q(4)+1}{4} = \frac{\frac{1}{6}(4-1)(4-2)(4-3)+1}{4} = \frac{\frac{1}{6} \cdot 3 \cdot 2 \cdot 1 + 1}{4} = \frac{2}{4} = \frac{1}{2}.$$

(It is also possible to show that $p(x) = \frac{1}{6}x^2 - x + \frac{11}{6}$.)

10. Consider a group of 6 people. Each of the six people knows exactly one piece of information, and all 6 pieces of information are different. Every time person "A" calls person "B" on the telephone, "A" tells "B" everything he or she knows, while "B" tells "A" nothing. What is the minimum number of telephone calls between pairs of people needed for everyone to know everything?

(20 pts) 10. 10 calls

Solution: Let p_i denote person i, $p_i \to p_j$ denote a call from p_i to p_j, and N denote the minimum number of calls which can leave everyone fully informed. The following sequence of calls leaves everyone informed:

$$p_1 \to p_2, \quad p_2 \to p_3, \quad p_3 \to p_4, \quad p_4 \to p_5, \quad p_5 \to p_6,$$
$$p_6 \to p_1, \quad p_1 \to p_2, \quad p_2 \to p_3, \quad p_3 \to p_4, \quad p_4 \to p_5.$$

This shows that $N \leq 10$.

We will now show that $N \geq 10$. Consider any sequence of calls which leaves everyone fully informed. Consider the "crucial" call at the end of which the receiver becomes the first person to know everything. Let r denote the receiver of the crucial call. Immediately after the crucial call, each of the other five people must have made at least one call. Otherwise, r would not know everything. So at least five calls have been made. But none of the other five people (other than r) knows everything. In order for each of the other five people to become fully informed, each of these other five people must receive at least one call. So at least five additional telephone calls must occur. But at least five calls have already occurred. So for all six people to know everything, a total of at least $5 + 5 = 10$ telephone calls must occur. So $N \geq 10$.

This shows that $N = 10$ calls is the minimum number for everyone to know everything.

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #2
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

1. Recall $i^2 = -1$. Simplify $i + i^2 + i^3 + i^4 + \cdots + i^{2023}$ as much as possible.

   (20 pts) 1. ________

2. What is the period of $y = \cos\left(\frac{2023x}{\pi^2}\right)$?

   (20 pts) 2. ________

3. Determine the diameter of the circle given by $4x^2 - 24x + 4y^2 - 16y = 48$.

   (20 pts) 3. ________

4. In terms of $\log x$, find the average of $\log x$, $\log x^2$, $\log x^3$, $\log x^4$, and $\log x^5$.

   (20 pts) 4. ________

5. Find the proportion of a circle's area which lies closer to its center than its boundary.

   (20 pts) 5. ________

6. Two bracelets are considered the same design if one can be flipped and rotated to match the other. If a bracelet is made using 6 different charms, how many possible designs are there for the bracelet?

   (20 pts) 6. ________

7. Let R be the rectangle bounded by the x-axis, y-axis, and lines $y = 2$ and $x = 3$. What is the probability that a randomly point $(x, y) \in R$ satisfies $1 < x + y < 3$?

   (20 pts) 7. ________

8. Assume today is Tuesday. What day of the week will it be in 2023 days?

   (20 pts) 8. ________

9. In the form $y = mx + b$, find the equation of line passing through (π^3, π^2) and perpendicular to $y = \frac{\pi}{2}x + \pi^3$.

   (20 pts) 9. ________

10. Traffic lights on a certain road are red with probability $\frac{2}{5}$. What is the probability that none of the next three traffic lights are red? Assume traffic lights are independent.

    (20 pts) 10. ________

TOTAL POINTS ________

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #2
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Key

1. Recall $i^2 = -1$. Simplify $i + i^2 + i^3 + i^4 + \cdots + i^{2023}$ as much as possible.

(20 pts) 1. -1

2. What is the period of $y = \cos\left(\frac{2023x}{\pi^2}\right)$?

(20 pts) 2. $\frac{2\pi^3}{2023}$

3. Determine the diameter of the circle given by $4x^2 - 24x + 4y^2 - 16y = 48$.

(20 pts) 3. 10

4. In terms of $\log x$, find the average of $\log x$, $\log x^2$, $\log x^3$, $\log x^4$, and $\log x^5$.

(20 pts) 4. $3 \log x$

5. Find the proportion of a circle's area which lies closer to its center than its boundary.

(20 pts) 5. $\frac{1}{4}$ or 0.25

6. Two bracelets are considered the same design if one can be flipped and rotated to match the other. If a bracelet is made using 6 different charms, how many possible designs are there for the bracelet?

(20 pts) 6. 60

7. Let R be the rectangle bounded by the x-axis, y-axis, and lines $y = 2$ and $x = 3$. What is the probability that a randomly point $(x, y) \in R$ satisfies $1 < x + y < 3$?

(20 pts) 7. $\frac{7}{12}$

8. Assume today is Tuesday. What day of the week will it be in 2023 days?

(20 pts) 8. Tuesday

9. In the form $y = mx + b$, find the equation of line passing through (π^3, π^2) and perpendicular to $y = \frac{\pi}{2}x + \pi^3$.

(20 pts) 9. $y = -\frac{2}{\pi}x + 3\pi^2$

10. Traffic lights on a certain road are red with probability $\frac{2}{5}$. What is the probability that none of the next three traffic lights are red? Assume traffic lights are independent.

(20 pts) 10. $\frac{27}{125}$ or 0.216

UND MATHEMATICS TRACK MEET
University of North Dakota
December 19, 2023

TEAM TEST #2
Grades 11/12

School ______________

Team Name ______________

Calculators are **NOT** allowed.

Solutions

1. Recall $i^2 = -1$. Simplify $i + i^2 + i^3 + i^4 + \cdots + i^{2023}$ as much as possible.

(20 pts) 1. -1

Solution: The key observation is that four consecutive powers of i sum to 0. For example, $i + i^2 + i^3 + i^4 = i - 1 - i + 1 = 0$. Thus

$$i + i^2 + i^3 + i^4 + \cdots + i^{2023} = i^{2021} + i^{2022} + i^{2023} = i - 1 - i = -1.$$

2. What is the period of $y = \cos\left(\frac{2023x}{\pi^2}\right)$?

(20 pts) 2. $\frac{2\pi^3}{2023}$

Solution: The period P of $y = \cos(Bx)$ is $P = \frac{2\pi}{B}$. It follows that the period of $y = \cos\left(\frac{2023x}{\pi^2}\right)$ is $P = \frac{2\pi}{2023/\pi^2} = \frac{2\pi^3}{2023}$.

3. Determine the diameter of the circle given by $4x^2 - 24x + 4y^2 - 16y = 48$.

(20 pts) 3. 10

Solution: Observe

$$\begin{aligned} 4x^2 - 24x + 4y^2 - 16y &= 48 \\ x^2 - 6x + y^2 - 4y &= 12 \\ (x-3)^2 + (y-2)^2 &= 12 + 9 + 4 == 25 \end{aligned}$$

Thus the radius of the circle is $r = \sqrt{25} = 5$ and so the diameter is $d = 2r = 2(5) = 10$.

4. In terms of $\log x$, find the average of $\log x$, $\log x^2$, $\log x^3$, $\log x^4$, and $\log x^5$.

(20 pts) 4. $3\log x$

Solution: The average is

$$\frac{\log x + \log x^2 + \log x^3 + \log x^4 + \log x^5}{5} = \frac{\log(x \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5)}{5} = \frac{\log(x^{15})}{5} = \frac{15\log x}{5} = 3\log x$$

5. Find the proportion of a circle's area which lies closer to its center than its boundary.

(20 pts) 5. $\frac{1}{4}$ or 0.25

Solution: For a circle with radius r, points with distance less than $r/2$ from the center lie closer to the center than its boundary. So, the proportion of a circle's area closer to its center than its boundary is $\frac{\pi(r/2)^2}{\pi r^2} = \frac{1}{4}$ or 0.25.

6. Two bracelets are considered the same design if one can be flipped and rotated to match the other. If a bracelet is made using 6 different charms, how many possible designs are there for the bracelet?

(20 pts) 6. 60

Solution: There are 5! ways to order 6 different charms. Each bracelet can be paired with its flipped version, so this gives $\frac{5!}{2} = \frac{120}{2} = 60$ designs.

7. Let R be the rectangle bounded by the x-axis, y-axis, and lines $y = 2$ and $x = 3$. What is the probability that a randomly point $(x, y) \in R$ satisfies $1 < x + y < 3$?

(20 pts) 7. $\frac{7}{12}$

Solution: The area of the rectangle is 6. The area below the region is $\frac{1}{2}$. The area above the region is 2. Therefore, the area of the region is $\frac{7}{2}$, and the probability of being in the region is $\dfrac{7/2}{6} = \frac{7}{12}$.

8. Assume today is Tuesday. What day of the week will it be in 2023 days?

(20 pts) 8. Tuesday

Solution: Since $2023 = 289 \cdot 7$, it follows that in 2023 days will be Tuesday.

9. In the form $y = mx + b$, find the equation of line passing through (π^3, π^2) and perpendicular to $y = \frac{\pi}{2}x + \pi^3$.

(20 pts) 9. $y = -\frac{2}{\pi}x + 3\pi^2$

Solution: Lines perpendicular to $y = \frac{\pi}{2}x + \pi^3$ have slope $-\frac{2}{\pi}$ and form $y = -\frac{2}{\pi}x + b$. Fitting to the point (π^3, π^2), we see that $b = \pi^2 + \frac{2}{\pi}(\pi^3) = \pi^2 + 2\pi^2 = 3\pi^2$. Thus the equation of the desired line is $y = -\frac{2}{\pi}x + 3\pi^2$.

10. Traffic lights on a certain road are red with probability $\frac{2}{5}$. What is the probability that none of the next three traffic lights are red? Assume traffic lights are independent.

(20 pts) 10. $\frac{27}{125}$ or 0.216

Solution: The probability a traffic light not being red is $1 - \frac{2}{5} = \frac{3}{5}$. Since the lights are independent, use the product rule for probability to conclude that the probability of no red lights in three traffic lights is $(\frac{3}{5})^3 = \frac{27}{125} = 0.216$.