APEX CALCULUS II
Late Transcendentals

University of North Dakota

Adapted from APEX Calculus by
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PREFACE

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text comprises a three-volume series on Calculus. The first part covers material taught in many "Calculus 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6. The second text covers material often taught in "Calculus 2": integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 7 through 10. The third text covers topics common in "Calculus 3" or "Multivariable Calculus": parametric equations, polar coordinates, vector-valued functions, and functions of more than one variable, found in Chapters 11 through 15. All three are available separately for free.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased separately.

A result of this splitting is that sometimes material is referenced that is not contained in the present text. The context should make it clear whether the "missing" material comes before or after the current portion. Downloading the appropriate pdf, or the entire *APEX Calculus LT* pdf, will give access to these topics.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High-level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower-level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just-learned material helps address future problems. Additionally, some chapters include a section zero, which provides a basic review and practice problems of pre-calculus skills. Since this content is a pre-requisite for calculus, reviewing and mastering these skills are considered your responsibility. This means that it is your responsibility to seek assistance outside of class from your instructor, a math resource center or other math tutoring available on-campus. A solid understanding of these skills is essential to your success in solving calculus problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Some proofs have been delayed until later (or omitted completely). In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the more difficult proofs in the text. The interested reader is highly encouraged to find other proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Work through the examples. The best way to learn mathematics is to do it. Reading about it (or watching someone else do it) is a poor substitute. For this reason, every page has a place for *you* to put *your* notes so that *you* can work out the examples. That being said, sometimes it is useful to watch someone work through an example. For this reason, this text also provides links to online videos where someone is working through a similar problem. If you want even more videos, these are generally chosen from

- Khan Academy: `https://www.khanacademy.org/`
- Math Doctor Bob: `http://www.mathdoctorbob.org/`
- Just Math Tutorials: `http://patrickjmt.com/` (unfortunately, they're not well organized)

Some other sites you may want to consider are

- Larry Green's Calculus Videos: `http://www.ltcconline.net/greenl/courses/105/videos/VideoIndex.htm`
- Mathispower4u: `http://www.mathispower4u.com/`
- Yay Math: `http://www.yaymath.org/` (for prerequisite material)

All of these sites are completely free (although some will ask you to donate). Here's a sample one:

Watch the video:
Practical Advice for Those Taking College Calculus
at
`https://youtu.be/ILNfpJTZLxk`

Thanks from Greg Hartman

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this "project from work" occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me

numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI's Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX — Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high-quality, low-cost textbooks. The current textbook-writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson-Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution — Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `https://github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

Greg Hartman

Creating APEX LT

Starting with the source at `https://github.com/APEXCalculus`, faculty at the University of North Dakota made several substantial changes to create APEX Late Transcendentals. The most obvious change was to rearrange the text to delay proving the derivative of transcendental functions until Calculus 2. UND added Sections 7.1 and 7.3, adapted several sections from other resources, created the prerequisite sections, included links to videos and Geogebra, and added several examples and exercises. In the end, every section had some changes (some more substantial than others), resulting in a document that is about 10% longer. The source files can now be found at
`https://github.com/teepeemm/APEXCalculusLT_Source`.

Extra thanks are due to Michael Corral for allowing us to use portions of his Vector Calculus, available at `www.mecmath.net/` (specifically, Section 13.9 and the Jacobian in Section 14.7) and to Paul Dawkins for allowing us to use portions of his online math notes from tutorial.math.lamar.edu/ (specifically, Sections 8.5 and 9.7, as well as "Area with Parametric Equations" in Section 10.3). The work on Calculus III was partially supported by the NDUS OER Initiative.

Electronic Resources

A distinctive feature of APEX is interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

Currently, the only pdf viewers that support these 3D graphics for computers are Adobe Reader & Acrobat. To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the "Disable Content" option.

The situation is more interesting for tablets and smartphones. The 3D graphics files have been arrayed at https://sites.und.edu/timothy.prescott/apex/prc/. At the bottom of the page are links to Android and iOS apps that can display the interactive files. The QR code to the right will take you to that page.

Additionally, a web version of the book is available at https://sites.und.edu/timothy.prescott/apex/web/. While we have striven to make the pdf accessible for non-print formats, html is far better in this regard.

Calculus II

7: Inverse Functions and L'Hôpital's Rule

This chapter completes our differentiation toolkit. The first and most important tool will be how to differentiate inverse functions. We'll be able to use this to differentiate exponential and logarithmic functions, which we stated in Theorem 2.3.1 but did not prove.

7.1 Inverse Functions

We say that two functions f and g are *inverses* if $g(f(x)) = x$ for all x in the domain of f and $f(g(x)) = x$ for all x in the domain of g. A function can only have an inverse if it is one-to-one, i.e. if we never have $f(x_1) = f(x_2)$ for different elements x_1 and x_2 of the domain. This is equivalent to saying that the graph of the function passes the horizontal line test. The inverse of f is denoted f^{-1}, which should not be confused with the function $1/f(x)$.

Key Idea 7.1.1 Inverse Functions

For a one-to–one function f,

- The domain of f^{-1} is the range of f; the range of f^{-1} is the domain of f.
- $f^{-1}(f(x)) = x$ for all x in the domain of f.
- $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
- The graph of $y = f^{-1}(x)$ is the reflection across $y = x$ of the graph of $y = f(x)$.
- $y = f^{-1}(x)$ if and only if $f(y) = x$ and y is in the domain of f.

Notes:

Watch the video:
Finding the Inverse of a Function or Showing One Does not Exist, Ex 3 at
https://youtu.be/BmjbDINGZGg

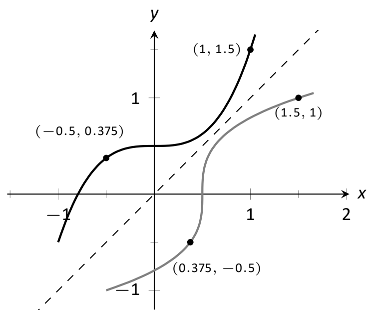


Figure 7.1.1: A function f along with its inverse f^{-1}. (Note how it does not matter which function we refer to as f; the other is f^{-1}.)

To determine whether or not f and g are inverses for each other, we check to see whether or not $g(f(x)) = x$ for all x in the domain of f,and $f(g(x)) = x$ for all x in the domain of g.

Example 7.1.1 Verifying Inverses

Determine whether or not the following pairs of functions are inverses:

1. $f(x) = 3x + 1$; $g(x) = \dfrac{x-1}{3}$
2. $f(x) = x^3 + 1$; $g(x) = x^{1/3} - 1$

Solution

1. To check the composition we plug $f(x)$ in for x in the definition of g as follows:
$$g(f(x)) = \frac{f(x)-1}{3} = \frac{(3x+1)-1}{3} = \frac{3x}{3} = x$$
So $g(f(x)) = x$ for all x in the domain of f. Likewise, you can check that $f(g(x)) = x$ for all x in the domain of g, so f and g are inverses.
2. If we try to proceed as before, we find that:
$$g(f(x)) = (f(x))^{1/3} - 1 = (x^3+1)^{1/3} - 1$$
This doesn't seem to be the same as the identity function x. To verify this, we find a number a in the domain of f and show that $g(f(a)) \neq a$ for that value. Let's try $x = 1$. Since $f(1) = 1^3 + 1 = 2$, we find that $g(f(1)) = g(2) = 2^{1/3} - 1 \approx 0.26$. Since $g(f(1)) \neq 1$, these functions are not inverses.

Functions that are not one-to-one.

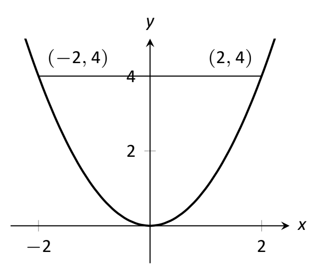


Figure 7.1.2: The function $f(x) = x^2$ is not one-to-one.

Unfortunately, not every function we would like to find an inverse for is one-to-one. For example, the function $f(x) = x^2$ is not one-to-one because $f(-2) = f(2) = 4$. If f^{-1} is an inverse for f, then $f^{-1}(f(-2)) = -2$ implies that $f^{-1}(4) =$

Notes:

-2. On the other hand, $f^{-1}(f(2)) = 2$, so $f^{-1}(4) = 2$. We cannot have it both ways if f^{-1} is a function, so no such inverse exists. We can find a partial solution to this dilemma by restricting the domain of f. There are many possible choices, but traditionally we restrict the domain to the interval $[0, \infty)$. The function $f^{-1}(x) = \sqrt{x}$ is now an inverse for this restricted version of f.

The inverse sine function

We consider the function $f(x) = \sin x$, which is not one-to-one. A piece of the graph of f is in Figure 7.1.3(a). In order to find an appropriate restriction of the domain of f, we look for consecutive critical points where f takes on its minimum and maximum values. In this case, we use the interval $[-\pi/2, \pi/2]$. We define the inverse of f on this restricted range by $y = \sin^{-1} x$ if and only if $\sin y = x$ and $-\pi/2 \leq y \leq \pi/2$. The graph is a reflection of the graph of g across the line $y = x$, as seen in Figure 7.1.3(b).

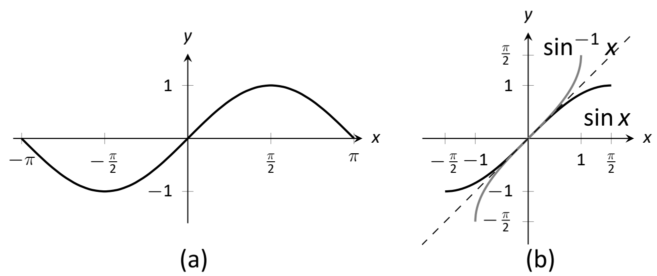


Figure 7.1.3: (a) A portion of $y = \sin x$. (b) A one-to-one portion of $y = \sin x$ along with $y = \sin^{-1} x$.

The inverse tangent function

Next we consider the function $f(x) = \tan x$, which is also not one-to-one. A piece of the graph of f is given in Figure 7.1.4(a). In order to find an interval on which the function is one-to-one and on which the function takes on all values in the range, we use an interval between consecutive vertical asymptotes. Traditionally, the interval $(-\pi/2, \pi/2)$ is chosen. Note that we choose the open interval in this case because the function f is not defined at the endpoints. So we define $y = \tan^{-1} x$ if and only if $\tan y = x$ and $-\pi/2 < y < \pi/2$. The graph of $y = \tan^{-1} x$ is shown in Figure 7.1.4(b). Also note that the vertical asymptotes of the original function are reflected to become horizontal asymptotes of the inverse function.

Notes:

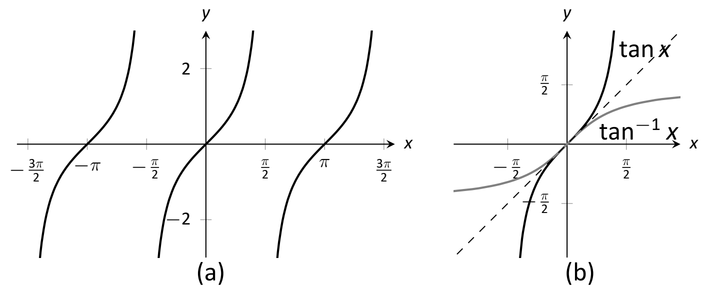


Figure 7.1.4: (a) A portion of $y = \tan x$. (b) A one-to-one portion of $y = \tan x$ along with $y = \tan^{-1}$.

The other inverse trigonometric functions are defined in a similar fashion. The resulting domains and ranges are summarized in Figure 7.1.5.

Function	Restricted Domain	Range	Inverse Function	Domain	Range
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

Figure 7.1.5: Domains and ranges of the trigonometric and inverse trigonometric functions.

Sometimes, arcsin is used instead of $\sin^{-1}$. Similar "arc" functions are used for the other inverse trigonometric functions as well.

Example 7.1.2 Evaluating Inverse Trigonometric Functions

Find exact values for the following:

1. $\tan^{-1}(1)$
2. $\cos(\sin^{-1}(\sqrt{3}/2))$
3. $\sin^{-1}(\sin(7\pi/6))$
4. $\tan(\cos^{-1}(11/15))$

Solution

1. $\tan^{-1}(1) = \pi/4$
2. $\cos(\sin^{-1}(\sqrt{3}/2)) = \cos(\pi/3) = 1/2$
3. Since $7\pi/6$ is not in the range of the inverse sine function, we should be careful with this one.

$$\sin^{-1}(\sin(7\pi/6)) = \sin^{-1}(-1/2) = -\pi/6.$$

Notes:

4. Since we don't know the value of $\cos^{-1}(11/15)$, we let θ stand for this value. We know that θ is an angle between 0 and π and that $\cos(\theta) = 11/15$. In Figure 7.1.6, we use this information to construct a right triangle with angle θ, where the adjacent side over the hypotenuse must equal 11/15. Applying the Pythagorean Theorem we find that

$$y = \sqrt{15^2 - 11^2} = \sqrt{104} = 2\sqrt{26}.$$

Finally, we have:

$$\tan(\cos^{-1}(11/15)) = \tan(\theta) = \frac{2\sqrt{26}}{11}.$$

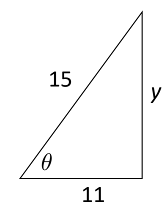


Figure 7.1.6: A right triangle for the situation in Example 7.1.2 (4).

Notes:

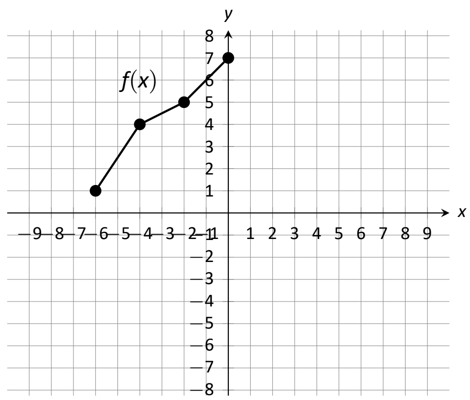
Exercises 7.1

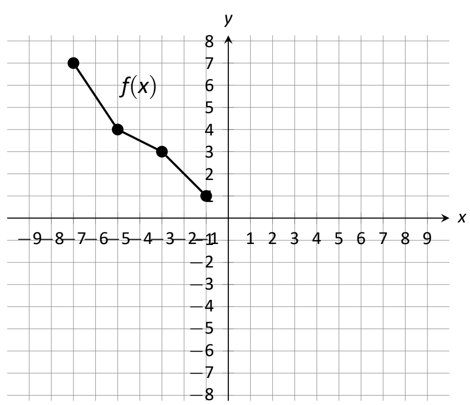
Terms and Concepts

1. T/F: Every function has an inverse.
2. In your own words explain what it means for a function to be “one to one.”
3. If $(1, 10)$ lies on the graph of $y = f(x)$, what can be said about the graph of $y = f^{-1}(x)$?
4. If a function doesn’t have an inverse, what can we do to help it have an inverse?

Problems

In Exercises 5–6, given the graph of f, sketch the graph of f^{-1}.

5. 

6. 

In Exercises 7–10, verify that the given functions are inverses.

7. $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$
8. $f(x) = x^2 + 6x + 11, x \geq -3$ and $g(x) = \sqrt{x-2} - 3, x \geq 2$
9. $f(x) = \dfrac{3}{x-5}, x \neq 5$ and $g(x) = \dfrac{3+5x}{x}, x \neq 0$
10. $f(x) = \dfrac{x+1}{x-1}, x \neq 1$ and $g(x) = f(x)$

In Exercises 11–14, find a restriction of the domain of the given function on which the function will have an inverse.

11. $f(x) = \sqrt{16 - x^2}$
12. $g(x) = \sqrt{x^2 - 16}$
13. $r(t) = t^2 - 6t + 9$
14. $f(x) = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

In Exercises 15–18, find the inverse of the given function.

15. $f(x) = \dfrac{x+1}{x-2}$
16. $f(x) = x^2 + 4$
17. $f(x) = e^{x+3} - 2$
18. $f(x) = \ln(x - 5) + 1$

In Exercises 19–28, find the exact value.

19. $\tan^{-1}(0)$
20. $\tan^{-1}(\tan(\pi/7))$
21. $\cos(\cos^{-1}(-1/5))$
22. $\sin^{-1}(\sin(8\pi/3))$
23. $\sin(\tan^{-1}(1))$
24. $\sec(\sin^{-1}(-3/5))$
25. $\cos(\tan^{-1}(3/7))$
26. $\sin^{-1}(-\sqrt{3}/2)$
27. $\cos^{-1}(-\sqrt{2}/2)$
28. $\cos^{-1}(\cos(8\pi/7))$

In Exercises 29–32, simplify the expression.

29. $\sin\left(\tan^{-1} \frac{x}{\sqrt{4-x^2}}\right)$
30. $\tan\left(\sin^{-1} \frac{x}{\sqrt{x^2+4}}\right)$
31. $\cos\left(\sin^{-1} \frac{5}{\sqrt{x^2+25}}\right)$
32. $\cot\left(\cos^{-1} \frac{3}{\sqrt{x}}\right)$
33. Show that for any x in the domain of $\sec^{-1}$ we have $\sec^{-1} x = \cos^{-1} \frac{1}{x}$.
34. Show that for $|x| \leq 1$ we have $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$. Hint: Recall the cofunction identity $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ for all θ.
35. Show that for any x we have $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$.
36. Show that for $|x| \geq 1$ we have $\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$.
37. A mass attached to a spring oscillates vertically about the equilibrium position $y = 0$ according to the function $y(t) = e^{-t}(\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t))$. Find the first positive time t for which $y(t) = 0$.

7.2 Derivatives of Inverse Functions

In this section we will figure out how to differentiate the inverse of a function. To do so, we recall that if f and g are inverses, then $f(g(x)) = x$ for all x in the domain of f. Differentiating and simplifying yields:

$$\begin{aligned} f(g(x)) &= x \\ f'(g(x))g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(g(x))} \quad \text{assuming } f'(x) \text{ is nonzero} \end{aligned}$$

Note that the derivation above assumes that the function g is differentiable. It is possible to prove that g must be differentiable if f' is nonzero, but the proof is beyond the scope of this text. However, assuming this fact we have shown the following:

Theorem 7.2.1 Derivatives of Inverse Functions
Let f be differentiable and one-to-one on an open interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

$$\begin{aligned} \left(f^{-1}\right)'(b) &= g'(b) = \frac{1}{f'(a)} \\ \left(f^{-1}\right)'(x) &= g'(x) = \frac{1}{f'(g(x))} \end{aligned}$$

The results of Theorem 7.2.1 are not trivial; the notation may seem confusing at first. Careful consideration, along with examples, should earn understanding.

Watch the video:
Derivative of an Inverse Function, Ex 2 at
`https://youtu.be/RKfGMXOpn2k`

In the next example we apply Theorem 7.2.1 to the arcsine function.

Notes:

Example 7.2.1 Finding the derivative of an inverse trigonometric function

Let $y = \sin^{-1} x$. Find y' using Theorem 7.2.1.

Solution Adopting our previously defined notation, let $g(x) = \sin^{-1} x$ and $f(x) = \sin x$. Thus $f'(x) = \cos x$. Applying Theorem 7.2.1, we have

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\sin^{-1} x)}. \end{aligned}$$

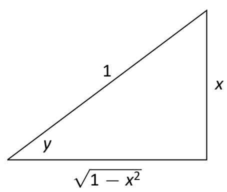


Figure 7.2.1: A right triangle defined by $y = \sin^{-1}(x/1)$ with the length of the third leg found using the Pythagorean Theorem.

This last expression is not immediately illuminating. Drawing a figure will help, as shown in Figure 7.2.1. Recall that the sine function can be viewed as taking in an angle and returning a ratio of sides of a right triangle, specifically, the ratio "opposite over hypotenuse." This means that the arcsine function takes as input a ratio of sides and returns an angle. The equation $y = \sin^{-1} x$ can be rewritten as $y = \sin^{-1}(x/1)$; that is, consider a right triangle where the hypotenuse has length 1 and the side opposite of the angle with measure y has length x. This means the final side has length $\sqrt{1-x^2}$, using the Pythagorean Theorem.

Therefore $\cos(\sin^{-1} x) = \cos y = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$, resulting in

$$\frac{d}{dx}\left(\sin^{-1} x\right) = g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Remember that the input x of the arcsine function is a ratio of a side of a right triangle to its hypotenuse; the absolute value of this ratio will be less than 1. Therefore $1 - x^2$ will be positive.

In order to make $y = \sin x$ one-to-one, we restrict its domain to $[-\pi/2, \pi/2]$; on this domain, the range is $[-1, 1]$. Therefore the domain of $y = \sin^{-1} x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. When $x = \pm 1$, note how the derivative of the arcsine function is undefined; this corresponds to the fact that as $x \to \pm 1$, the tangent lines to arcsine approach vertical lines with undefined slopes.

In Figure 7.2.2 we see $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$ graphed on their respective domains. The line tangent to $\sin x$ at the point $(\pi/3, \sqrt{3}/2)$ has slope $\cos \pi/3 = 1/2$. The slope of the corresponding point on $\sin^{-1} x$, the point $(\sqrt{3}/2, \pi/3)$, is

$$\frac{1}{\sqrt{1-(\sqrt{3}/2)^2}} = \frac{1}{\sqrt{1-3/4}} = \frac{1}{\sqrt{1/4}} = \frac{1}{1/2} = 2,$$

verifying Theorem 7.2.1 yet again: at corresponding points, a function and its inverse have reciprocal slopes.

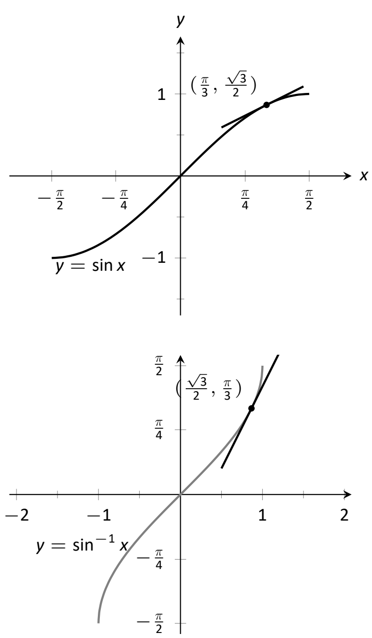


Figure 7.2.2: Graphs of $y = \sin x$ and $y = \sin^{-1} x$ along with corresponding tangent lines.

Notes:

Using similar techniques, we can find the derivatives of all the inverse trigonometric functions after first restricting their domains according to Figure 7.1.5 to allow them to be invertible.

Theorem 7.2.2 Derivatives of Inverse Trigonometric Functions
The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Figure 7.1.5) and their derivatives are as follows:

1. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = -\dfrac{1}{\sqrt{1-x^2}}$
5. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}$
6. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = -\dfrac{1}{1+x^2}$

Note how the last three derivatives are merely the negatives of the first three, respectively. Because of this, the first three are used almost exclusively throughout this text.

Example 7.2.2 Finding derivatives of inverse functions
Find the derivatives of the following functions:

1. $f(x) = \cos^{-1}(x^2)$ 2. $g(x) = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ 3. $f(x) = \sin^{-1}(\cos x)$

Solution

1. We use Theorem 7.2.2 and the Chain Rule to find:

$$f'(x) = -\frac{1}{\sqrt{1-(x^2)^2}}(2x) = -\frac{2x}{\sqrt{1-x^4}}$$

2. We use Theorem 7.2.2 and the Quotient Rule to compute:

$$\begin{aligned} g'(x) &= \frac{\left(\frac{1}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} - (\sin^{-1} x)\left(\frac{1}{2\sqrt{1-x^2}}(-2x)\right)}{\left(\sqrt{1-x^2}\right)^2} \\ &= \frac{\sqrt{1-x^2} + x\sin^{-1} x}{\left(\sqrt{1-x^2}\right)^3} \end{aligned}$$

Notes:

3. We apply Theorem 7.2.2 and the Chain Rule again to compute:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-\cos^2 x}}(-\sin x) \\ &= \frac{-\sin x}{\sqrt{\sin^2 x}} \\ &= \frac{-\sin x}{|\sin x|}. \end{aligned}$$

Theorem 7.2.2 allows us to integrate some functions that we could not integrate before. For example,

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$$

Combining these formulas with u-substitution yields the following:

Theorem 7.2.3 Integrals Involving Inverse Trigonometric Functions

Let $a > 0$. Then

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

We will look at the second part of this theorem. The other parts are similar and are left as exercises.

First we note that the integrand involves the number a^2, but does not explicitly involve a. We make the assumption that $a > 0$ in order to simplify what follows. We can rewrite the integral as follows:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{\sqrt{a^2(1-(x/a)^2)}} = \int \frac{dx}{a\sqrt{1-(x/a)^2}}$$

Notes:

We next use the substitution $u = x/a$ and $\mathrm{d}u = \mathrm{d}x/a$ to find:

$$\begin{aligned}\int \frac{\mathrm{d}x}{a\sqrt{1-(x/a)^2}} &= \int \frac{a}{a\sqrt{1-u^2}}\,\mathrm{d}u\\ &= \int \frac{\mathrm{d}u}{\sqrt{1-u^2}}\\ &= \sin^{-1} u + C\\ &= \sin^{-1}(x/a) + C\end{aligned}$$

We conclude this section with several examples.

Example 7.2.3 Finding antiderivatives involving inverse functions

Find the following integrals.

1. $\displaystyle\int \frac{\mathrm{d}x}{100+x^2}$ 2. $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,\mathrm{d}x$ 3. $\displaystyle\int \frac{\mathrm{d}x}{x^2+2x+5}$

Solution

1. $\displaystyle\int \frac{\mathrm{d}x}{100+x^2} = \int \frac{\mathrm{d}x}{10^2+x^2} = \frac{1}{10}\tan^{-1}(x/10) + C$

2. We use the substitution $u = \sin^{-1} x$ and $\mathrm{d}u = \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ to find:

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\sin^{-1} x\right)^2 + C$$

3. This does not immediately look like one of the forms in Theorem 7.2.3, but we can complete the square in the denominator to see that

$$\int \frac{\mathrm{d}x}{x^2+2x+5} = \int \frac{\mathrm{d}x}{(x^2+2x+1)+4} = \int \frac{\mathrm{d}x}{4+(x+1)^2}$$

We now use the substitution $u = x+1$ and $\mathrm{d}u = \mathrm{d}x$ to find:

$$\begin{aligned}\int \frac{\mathrm{d}x}{4+(x+1)^2} &= \int \frac{\mathrm{d}u}{4+u^2}\\ &= \frac{1}{2}\tan^{-1}(u/2) + C = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.\end{aligned}$$

Notes:

Exercises 7.2

Terms and Concepts

1. If $(1, 10)$ lies on the graph of $y = f(x)$ and $f'(1) = 5$, what can be said about $y = f^{-1}(x)$?
2. Since $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$, what does this tell us about $\sin^{-1} x + \cos^{-1} x$?

Problems

In Exercises 3–8, an invertible function $f(x)$ is given along with a point that lies on its graph. Using Theorem 7.2.1, evaluate $\left(f^{-1}\right)'(x)$ at the indicated value.

3. $f(x) = 5x + 10$
   Point$= (2, 20)$
   Evaluate $\left(f^{-1}\right)'(20)$
4. $f(x) = x^2 - 2x + 4$, $x \geq 1$
   Point$= (3, 7)$
   Evaluate $\left(f^{-1}\right)'(7)$
5. $f(x) = \sin 2x$, $-\pi/4 \leq x \leq \pi/4$
   Point$= (\pi/6, \sqrt{3}/2)$
   Evaluate $\left(f^{-1}\right)'(\sqrt{3}/2)$
6. $f(x) = x^3 - 6x^2 + 15x - 2$
   Point$= (1, 8)$
   Evaluate $\left(f^{-1}\right)'(8)$
7. $f(x) = \dfrac{1}{1 + x^2}$, $x \geq 0$
   Point$= (1, 1/2)$
   Evaluate $\left(f^{-1}\right)'(1/2)$
8. $f(x) = 6e^{3x}$
   Point$= (0, 6)$
   Evaluate $\left(f^{-1}\right)'(6)$

In Exercises 9–18, compute the derivative of the given function.

9. $h(t) = \sin^{-1}(2t)$
10. $f(t) = \sec^{-1}(2t)$
11. $g(x) = \tan^{-1}(2x)$
12. $f(x) = x \sin^{-1} x$
13. $g(t) = \sin t \cos^{-1} t$
14. $f(t) = e^t \ln t$
15. $h(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$
16. $g(x) = \tan^{-1}(\sqrt{x})$
17. $f(x) = \sec^{-1}(1/x)$
18. $f(x) = \sin(\sin^{-1} x)$

In Exercises 19–22, compute the derivative of the given function in two ways:

(a) By simplifying first, then taking the derivative, and
(b) by using the Chain Rule first then simplifying.

Verify that the two answers are the same.

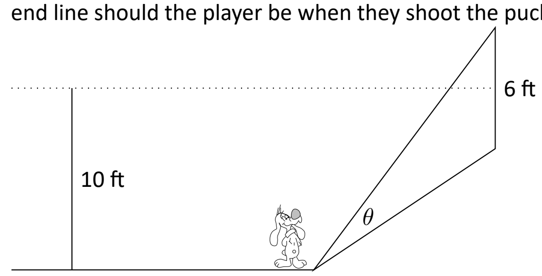
19. $f(x) = \sin(\sin^{-1} x)$
20. $f(x) = \tan^{-1}(\tan x)$
21. $f(x) = \sin(\cos^{-1} x)$
22. $f(x) = \sin(\tan^{-1} x)$

In Exercises 23–24, find the equation of the line tangent to the graph of f at the indicated x value.

23. $f(x) = \sin^{-1} x \quad$ at $\quad x = \frac{\sqrt{2}}{2}$
24. $f(x) = \cos^{-1}(2x) \quad$ at $\quad x = \frac{\sqrt{3}}{4}$

In Exercises 25–30, compute the indicated integral.

25. $\displaystyle\int_{1/\sqrt{2}}^{1/2} \frac{2}{\sqrt{1 - x^2}}\, dx$
26. $\displaystyle\int_0^{\sqrt{3}} \frac{4}{9 + x^2}\, dx$
27. $\displaystyle\int \frac{\sin^{-1} r}{\sqrt{1 - r^2}}\, dr$
28. $\displaystyle\int \frac{x^3}{4 + x^8}\, dx$
29. $\displaystyle\int \frac{e^t}{\sqrt{10 - e^{2t}}}\, dt$
30. $\displaystyle\int \frac{1}{\sqrt{x}(1 + x)}\, dx$
31. A regulation hockey goal is 6 feet wide. If a player is skating towards the end line on a line perpendicular to the end line and 10 feet from the imaginary line joining the center of one goal to the center of the other, the angle between the player and the goal first increases and then begins to decrease. In order to maximize this angle, how far from the end line should the player be when they shoot the puck?



7.3 Exponential and Logarithmic Functions

In this section we will define general exponential and logarithmic functions and find their derivatives.

General exponential functions

Consider first the function $f(x) = 2^x$. If x is rational, then we know how to compute 2^x. What do we mean by 2^π though? We compute this by first looking at 2^r for rational numbers r that are very close to π, then finding a limit. In our case we might compute 2^3, $2^{3.1}$, $2^{3.14}$, etc. We then define 2^π to be the limit of these numbers. Note that this is actually a different kind of limit than we have dealt with before since we only consider rational numbers close to π, not all real numbers close to π. We will see one way to make this more precise in Chapter 9. Graphically, we can plot the values of 2^x for x rational and get something like the dotted curve in Figure 7.3.1. In order to define the remaining values, we are "connecting the dots" in a way that makes the function continuous.

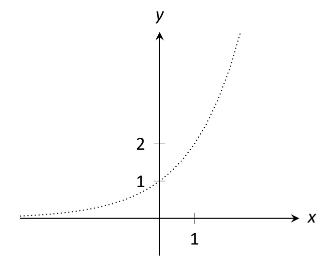


Figure 7.3.1: The function 2^x for rational values of x.

It follows from continuity and the properties of limits that exponential functions will satisfy the familiar properties of exponents (see Section 2.0). This implies that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x},$$

so the graph of $g(x) = (1/2)^x$ is the reflection of f across the y-axis, as in Figure 7.3.2.

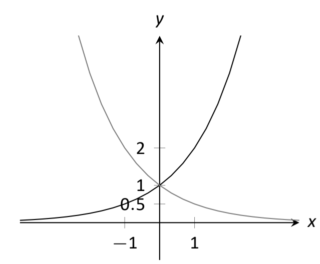


Figure 7.3.2: The functions 2^x and 2^{-x}.

We can go through the same process as above for any base $a > 0$, though we are not usually interested in the constant function 1^x.

Key Idea 7.3.1 Properties of Exponential Functions

For $a > 0$ and $a \neq 1$ the exponential function $f(x) = a^x$ satisfies:

1. $a^0 = 1$
2. $\displaystyle\lim_{x\to\infty} a^x = \begin{cases} \infty & a > 1 \\ 0 & a < 1 \end{cases}$
3. $a^x > 0$ for all x
4. $\displaystyle\lim_{x\to-\infty} a^x = \begin{cases} 0 & a > 1 \\ \infty & a < 1 \end{cases}$

Notes:

Derivatives of exponential functions

Suppose $f(x) = a^x$ for some $a > 0$. We can use the rules of exponents to find the derivative of f:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \qquad \text{(since } a^x \text{ does not depend on } h\text{)} \end{aligned}$$

So we know that $f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$, but can we say anything about that remaining limit? First we note that

$$f'(0) = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h},$$

so we have $f'(x) = a^x f'(0)$. The actual value of the limit $\lim_{h\to 0} \frac{a^h - 1}{h}$ depends on the base a, but it can be proved that it does exist. We will figure out just what this limit is later, but for now we note that the easiest differentiation formulas come from using a base a that makes $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$. This base is the number $e \approx 2.71828$ and the exponential function e^x is called the natural exponential function. This leads to the following result.

Theorem 7.3.1 Derivative of Exponential Functions
For any base $a > 0$, the exponential function $f(x) = a^x$ has derivative $f'(x) = a^x f'(0)$. The natural exponential function $g(x) = e^x$ has derivative $g'(x) = e^x$.

Notes:

Watch the video:
Derivatives of Exponential Functions at
https://youtu.be/U3PyUcEd7IU

General logarithmic functions

Before reviewing general logarithmic functions, we'll first remind ourselves of the laws of logarithms.

Key Idea 7.3.2 Properties of Logarithms

For $a, x, y > 0$ and $a \neq 1$, we have

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$
3. $\log_x y = \dfrac{\log_a y}{\log_a x}$, when $x \neq 1$
4. $\log_a x^y = y \log_a x$
5. $\log_a 1 = 0$
6. $\log_a a = 1$

Let us consider the function $f(x) = a^x$ where $a \neq 1$. We know that $f'(x) = f'(0)a^x$, where $f'(0)$ is a constant that depends on the base a. Since $a^x > 0$ for all x, this implies that $f'(x)$ is either always positive or always negative, depending on the sign of $f'(0)$. This in turn implies that f is strictly monotonic, so f is one-to-one. We can now say that f has an inverse. We call this inverse the logarithm with base a, denoted $f^{-1}(x) = \log_a x$. When $a = e$, this is the natural logarithm function $\ln x$. So we can say that $y = \log_a x$ if and only if $a^y = x$. Since the range of the exponential function is the set of positive real numbers, the domain of the logarithm function is also the set of positive real numbers. Reflecting the graph of $y = a^x$ across the line $y = x$ we find that (for $a > 1$) the graph of the logarithm looks like Figure 7.3.3.

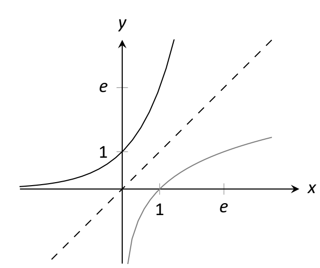


Figure 7.3.3: The functions $y = a^x$ and $y = \log_a x$ for $a > 1$.

Notes:

Key Idea 7.3.3 Properties of Logarithmic Functions
For $a > 0$ and $a \neq 1$ the logarithmic function $f(x) = \log_a x$ satisfies:

1. The domain of $f(x) = log_a x$ is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. $y = \log_a x$ if and only if $a^y = x$.
3. $\lim\limits_{x\to\infty} \log_a x = \begin{cases} \infty & \text{if } a > 1 \\ -\infty & \text{if } a < 1 \end{cases}$
4. $\lim\limits_{x\to 0^+} \log_a x = \begin{cases} -\infty & \text{if } a > 1 \\ \infty & \text{if } a < 1 \end{cases}$

Using the inverse of the natural exponential function, we can determine what the value of $f'(0)$ is in the formula $(a^x)' = f'(0)a^x$. To do so, we note that $a = e^{\ln a}$ since the exponential and logarithm functions are inverses. Hence we can write:

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

Now since $\ln a$ is a constant, we can use the Chain Rule to see that:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}(\ln a) = a^x \ln a$$

Comparing this to our previous result, we can restate our theorem:

Theorem 7.3.2 Derivative of Exponential Functions
For any base $a > 0$, the exponential function $f(x) = a^x$ has derivative $f'(x) = a^x \ln a$. The natural exponential function $g(x) = e^x$ has derivative $g'(x) = e^x$.

Change of base

In the previous computation, we found it convenient to rewrite the general exponential function in terms of the natural exponential function. A related formula

Notes:

allows us to rewrite the general logarithmic function in terms of the natural logarithm. To see how this works, suppose that $y = \log_a x$, then we have:

$$\begin{aligned} a^y &= x \\ \ln(a^y) &= \ln x \\ y \ln a &= \ln x \\ y &= \frac{\ln x}{\ln a} \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

This change of base formula allows us to use facts about the natural logarithm to derive facts about the general logarithm.

Derivatives of logarithmic functions

Since the natural logarithm function is the inverse of the natural exponential function, we can use the formula $(f^{-1}(x))' = \dfrac{1}{f'(f^{-1}(x))}$ to find the derivative of $y = \ln x$. We know that $\dfrac{d}{dx}e^x = e^x$, so we get:

$$\frac{d}{dx}\ln x = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Now we can apply the change of base formula to find the derivative of a general logarithmic function:

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\left(\frac{d}{dx}\ln x\right) = \frac{1}{x \ln a}.$$

Example 7.3.1 Finding Derivatives of Logs and Exponentials

Find derivatives of the following functions.

1. $f(x) = x3^{4x-7}$ 2. $g(x) = 2^{x^2}$ 3. $h(x) = \dfrac{x}{\log_5 x}$

Solution

1. We apply both the Product and Chain Rules:

$$f'(x) = 3^{4x-7} + x\left(3^{4x-7}\ln 3\right)(4) = (1 + 4x\ln 3)3^{4x-7}$$

2. We apply the Chain Rule:

$$g'(x) = 2^{x^2}(\ln 2)(2x) = 2^{x^2+1}x\ln 2.$$

Notes:

3. Applying the Quotient Rule:

$$h'(x) = \frac{\log_5 x - x\left(\frac{1}{x \ln 5}\right)}{(\log_5 x)^2} = \frac{(\log_5 x)(\ln 5) - 1}{(\log_5 x)^2 \ln 5}$$

Example 7.3.2 The Derivative of the Natural Log

Find the derivative of the function $y = \ln |x|$.

Solution We can rewrite our function as

$$y = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

Applying the Chain Rule, we see that $\frac{dy}{dx} = \frac{1}{x}$ for $x > 0$, and $\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$ for $x < 0$. Hence we have

$$\frac{d}{dx} \ln |x| = \frac{1}{x} \quad \text{for } x \neq 0.$$

Antiderivatives

Combining these new results, we arrive at the following theorem:

Theorem 7.3.3 Derivatives and Antiderivatives of Exponentials and Logarithms

Given a base $a > 0$ and $a \neq 1$, the following hold:

1. $\frac{d}{dx} e^x = e^x$
2. $\frac{d}{dx} a^x = a^x \ln a$
3. $\frac{d}{dx} \ln x = \frac{1}{x}$
4. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$
5. $\int e^x \, dx = e^x + C$
6. $\int a^x \, dx = \frac{a^x}{\ln a} + C$
7. $\int \frac{dx}{x} = \ln |x| + C$

Notes:

Example 7.3.3 Finding Antiderivatives

Find the following antiderivatives.

$$1. \int 3^x \, dx \qquad 2. \int x^2 e^{x^3} \, dx \qquad 3. \int \frac{x \, dx}{x^2+1}$$

Solution

1. Applying our theorem,

$$\int 3^x \, dx = \frac{3^x}{\ln 3} + C$$

2. We use the substitution $u = x^3$, $du = 3x^2 \, dx$:

$$\begin{aligned} \int x^2 e^{x^3} \, dx &= \frac{1}{3} \int e^u \, du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{x^3} + C \end{aligned}$$

3. Using the substitution $u = x^2 + 1$, $du = 2x \, dx$:

$$\begin{aligned} \int \frac{x \, dx}{x^2+1} &= \frac{1}{2} \int \frac{du}{u} \\ &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln \left| x^2 + 1 \right| + C \\ &= \frac{1}{2} \ln(x^2+1) + C \end{aligned}$$

Note that we do not yet have an antiderivative for the function $f(x) = \ln x$. We remedy this in Section 8.1 with Example 8.1.5.

Logarithmic Differentiation

Consider the function $y = x^x$; it is graphed in Figure 7.3.4. It is well-defined for $x > 0$ and we might be interested in finding equations of lines tangent and normal to its graph. How do we take its derivative?

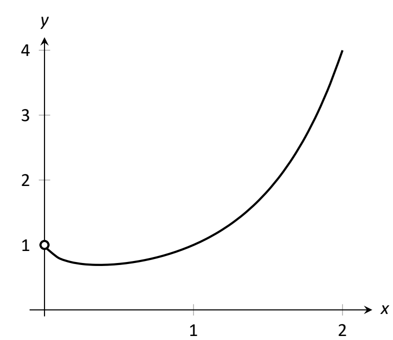


Figure 7.3.4: A plot of $y = x^x$.

Notes:

The function is not a power function: it has a "power" of x, not a constant. It is not an exponential function: it has a "base" of x, not a constant.

A differentiation technique known as *logarithmic differentiation* becomes useful here. The basic principle is this: take the natural log of both sides of an equation $y = f(x)$, then use implicit differentiation to find y'. We demonstrate this in the following example.

Example 7.3.4 Using Logarithmic Differentiation

Given $y = x^x$, use logarithmic differentiation to find y'.

Solution As suggested above, we start by taking the natural log of both sides then applying implicit differentiation.

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln(x^x) && \text{(apply logarithm rule)} \\
\ln(y) &= x\ln x && \text{(now use implicit differentiation)} \\
\frac{d}{dx}\Big(\ln(y)\Big) &= \frac{d}{dx}\Big(x\ln x\Big) \\
\frac{y'}{y} &= \ln x + x\cdot\frac{1}{x} \\
\frac{y'}{y} &= \ln x + 1 \\
y' &= y\big(\ln x + 1\big) && \text{(substitute } y = x^x\text{)} \\
y' &= x^x\big(\ln x + 1\big).
\end{aligned}$$

To "test" our answer, let's use it to find the equation of the tangent line at $x = 1.5$. The point on the graph our tangent line must pass through is $(1.5, 1.5^{1.5}) \approx (1.5, 1.837)$. Using the equation for y', we find the slope as

$$y' = 1.5^{1.5}\big(\ln 1.5 + 1\big) \approx 1.837(1.405) \approx 2.582.$$

Thus the equation of the tangent line is $y = 2.582(x - 1.5) + 1.837$. Figure 7.3.5 graphs $y = x^x$ along with this tangent line.

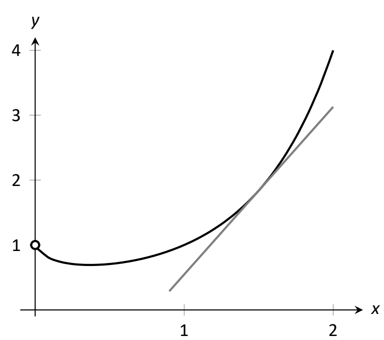


Figure 7.3.5: A graph of $y = x^x$ and its tangent line at $x = 1.5$.

Notes:

Exercises 7.3

Problems

In Exercises 1–4, find the domain of the function.

1. $f(x) = e^{x^2+1}$
2. $f(t) = \ln(1 - t^2)$
3. $g(x) = \ln(x^2)$
4. $f(x) = \dfrac{2}{\log_3(x^2 + 1)}$

In Exercises 5–12, find the derivative of the function.

5. $f(t) = e^{t^3-1}$
6. $g(r) = r^2 \log_2 r$
7. $f(x) = \dfrac{\log_5 x}{5^x}$
8. $f(x) = 4^{x^5}$
9. $f(x) = 7^{\log_7 x}$
10. $g(x) = e^{x^2} \sin(x - \ln x)$
11. $h(r) = \tan^{-1}(3^r)$
12. $h(x) = \log_{10}\left(\dfrac{x^2 + 1}{x^4}\right)$

In Exercises 13–24, evaluate the integral.

13. $\displaystyle\int_0^2 5^x \, dx$
14. $\displaystyle\int_1^3 \frac{\log_3 x}{x} \, dx$
15. $\displaystyle\int x3^{x^2-1} \, dx$
16. $\displaystyle\int \frac{\cos(\ln x)}{x} \, dx$
17. $\displaystyle\int e^x \sin(e^x) \cos(e^x) \, dx$
18. $\displaystyle\int_1^8 \log_2 x \, dx$
19. $\displaystyle\int_0^5 \frac{3^x}{3^x + 2} \, dx$
20. $\displaystyle\int \frac{1}{(1 + x^2)\tan^{-1} x} \, dx$
21. $\displaystyle\int \frac{\ln x}{x} \, dx$
22. $\displaystyle\int \frac{(\ln x)^2}{x} \, dx$
23. $\displaystyle\int \frac{\ln (x^3)}{x} \, dx$
24. $\displaystyle\int \frac{1}{x \ln (x^2)} \, dx$
25. Find the two values of n so that the function $y = e^{nx}$ satisfies the differential equation $y'' + y' - 6y = 0$.
26. Let $f(x) = x^2$ and $g(x) = 2^x$.
    (a) Since $f(2) = 2^2 = 4$ and $g(2) = 2^2 = 4$, $f(2) = g(2)$. Find a positive number $c > 2$ so that $f(c) = g(c)$.
    (b) Explain how you can be sure that there is at least one negative number a so that $f(a) = g(a)$.
    (c) Use the Bisection Method to estimate the number a accurate to within .05.
    (d) Assume you were to graph $f(x)$ and $g(x)$ on the same graph with unit length equal to 1 inch along both coordinate axes. Approximately how high is the graph of f when $x = 18$? The graph of g?

In Exercises 27–34, use logarithmic differentiation to find $\dfrac{dy}{dx}$, then find the equation of the tangent line at the indicated x-value.

27. $y = (1 + x)^{1/x}, \quad x = 1$
28. $y = (2x)^{x^2}, \quad x = 1$
29. $y = \dfrac{x^x}{x + 1}, \quad x = 1$
30. $y = x^{\sin(x)+2}, \quad x = \pi/2$
31. $y = \dfrac{x + 1}{x + 2}, \quad x = 1$
32. $y = \dfrac{(x + 1)(x + 2)}{(x + 3)(x + 4)}, \quad x = 0$
33. $y = x^{e^x}, x = 1$
34. $y = (\cot x)^{\cos x}, x = \pi$
35. The amount y of C^{14} in an object decays according the the function $y(t) = y_0 e^{-rt}$, where y_0 denotes the initial amount. If it takes 5730 years for half the initial amount to decay, find the rate constant r, and then determines how long it takes until only 10% remains.

7.4 Hyperbolic Functions

The **hyperbolic functions** are functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many connections between them and the standard trigonometric functions. Figure 7.4.1 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2 + y^2 = 1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2 - y^2 = 1$.

We begin with their definitions.

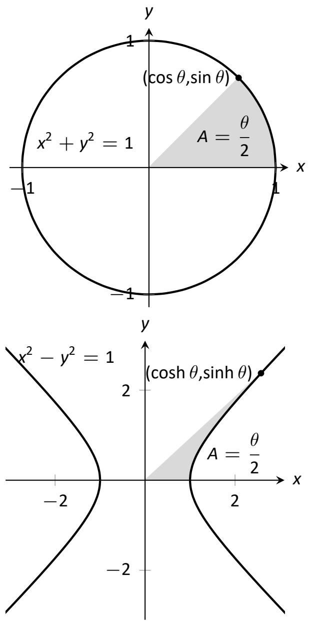


Figure 7.4.1: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola.

Definition 7.4.1 Hyperbolic Functions

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

Pronunciation Note:
"cosh" rhymes with "gosh,"
"sinh" rhymes with "pinch," and
"tanh" rhymes with "ranch,"

The hyperbolic functions are graphed in Figure 7.4.2. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \to \infty$, both $\sinh x$ and $\cosh x$ approach $e^x/2$, hence $\tanh x$ approaches 1.

Notes:

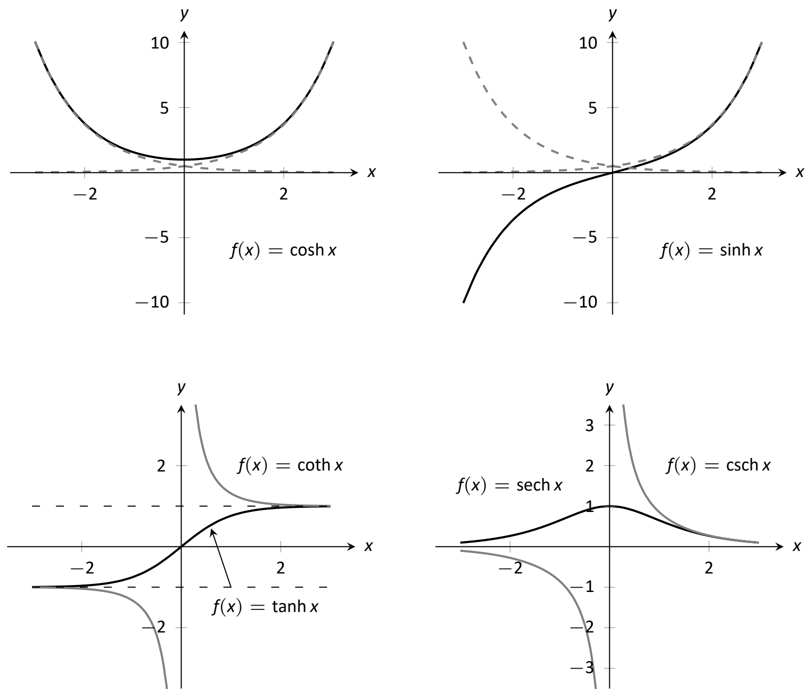


Figure 7.4.2: Graphs of the hyperbolic functions.

Watch the video:
Hyperbolic Functions — The Basics at
`https://youtu.be/G1C1Z5aTZSQ`

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Example 7.4.1 Exploring properties of hyperbolic functions

Use Definition 7.4.1 to rewrite the following expressions.

Notes:

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\big(\cosh x\big)$
5. $\frac{d}{dx}\big(\sinh x\big)$
6. $\frac{d}{dx}\big(\tanh x\big)$

Solution

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} \qquad \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)\\ &= 2\cdot\frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^x - e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

Notes:

5.
$$\begin{aligned}\frac{d}{dx}\big(\sinh x\big) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

6.
$$\begin{aligned}\frac{d}{dx}\big(\tanh x\big) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

So $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$.

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 7.4.1.

Key Idea 7.4.1 Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2 \sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
2. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}\big(\operatorname{sech} x\big) = -\operatorname{sech} x \tanh x$
5. $\frac{d}{dx}\big(\operatorname{csch} x\big) = -\operatorname{csch} x \coth x$
6. $\frac{d}{dx}\big(\coth x\big) = -\operatorname{csch}^2 x$

Integrals

1. $\displaystyle\int \cosh x\, dx = \sinh x + C$
2. $\displaystyle\int \sinh x\, dx = \cosh x + C$
3. $\displaystyle\int \tanh x\, dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + C$

We practice using Key Idea 7.4.1.

Notes:

Example 7.4.2 Derivatives and integrals of hyperbolic functions

Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \operatorname{sech}^2(7t-3)\,dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\,dx$

Solution

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.

   Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 7.4.1: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

   $$\begin{aligned}\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x\sinh x + 2\sinh x\cosh x\\ &= 4\cosh x\sinh x.\end{aligned}$$

   Using another Basic Identity, we can see that $4\cosh x\sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t-3$ and $du = 7\,dt$. Applying Key Idea 7.4.1 we have:

   $$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3. $$\int_0^{\ln 2}\cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

   We can simplify this last expression as $\sinh x$ is based on exponentials:

   $$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-1/2}{2} = \frac{3}{4}.$$

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 7.4.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 7.4.4.

Notes:

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 7.4.2. It is often more convenient to use $\sinh^{-1} x$ than $\ln\left(x + \sqrt{x^2+1}\right)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

Function	Domain	Range	Function	Domain	Range
$\cosh x$	$[0,\infty)$	$[1,\infty)$	$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\sinh x$	$(-\infty,\infty)$	$(-\infty,\infty)$	$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh x$	$(-\infty,\infty)$	$(-1,1)$	$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\operatorname{sech} x$	$[0,\infty)$	$(0,1]$	$\operatorname{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\operatorname{csch} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$	$\operatorname{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,-1)\cup(1,\infty)$	$\coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

Figure 7.4.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

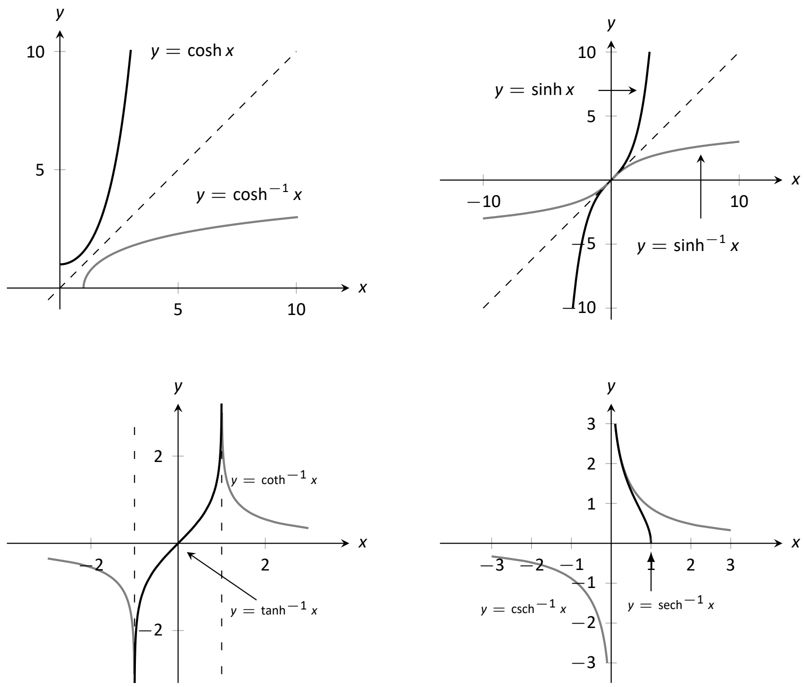


Figure 7.4.4: Graphs of the hyperbolic functions and their inverses.

Notes:

Now let's consider the inverses of the hyperbolic functions. We begin with the function $f(x) = \sinh x$. Since $f'(x) = \cosh x > 0$ for all real x, f is increasing and must be one-to-one.

$$\begin{aligned} y &= \frac{e^x - e^{-x}}{2} \\ 2y &= e^x - e^{-x} \qquad \text{(now multiply by } e^x\text{)} \\ 2ye^x &= e^{2x} - 1 \qquad \text{(a quadratic form)} \\ (e^x)^2 - 2ye^x - 1 &= 0 \qquad \text{(use the quadratic formula)} \\ e^x &= \frac{2y \pm \sqrt{4y^2+4}}{2} \\ e^x &= y \pm \sqrt{y^2+1} \qquad \text{(use the fact that } e^x > 0\text{)} \\ e^x &= y + \sqrt{y^2+1} \\ x &= \ln(y + \sqrt{y^2+1}) \end{aligned}$$

Finally, interchange the variable to find that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

In a similar manner we find that the inverses of the other hyperbolic functions are given by:

Key Idea 7.4.2 Logarithmic definitions of Inverse Hyperbolic Functions

1. $\cosh^{-1} x = \ln\big(x + \sqrt{x^2-1}\big)$; $x \geq 1$
2. $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$; $|x| < 1$
3. $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$; $0 < x \leq 1$
4. $\sinh^{-1} x = \ln\big(x + \sqrt{x^2+1}\big)$
5. $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$; $|x| > 1$
6. $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{|x|}\right)$; $x \neq 0$

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 7.4.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 7.4.2. Again, these latter functions are often more useful than the former.

Notes:

Key Idea 7.4.3 Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}}$; $x > 1$
2. $\dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$
3. $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$; $|x| < 1$
4. $\dfrac{d}{dx}\left(\text{sech}^{-1} x\right) = \dfrac{-1}{x\sqrt{1-x^2}}$; $0 < x < 1$
5. $\dfrac{d}{dx}\left(\text{csch}^{-1} x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}}$; $x \neq 0$
6. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2}$; $|x| > 1$

Key Idea 7.4.4 Integrals Involving Inverse Hyperbolic Functions

1. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C;\ 0 < a < x \quad = \ln\left(x+\sqrt{x^2-a^2}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C;\ a > 0 \quad = \ln\left(x+\sqrt{x^2+a^2}\right) + C$
3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C & |x| < |a| \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C & |a| < |x| \end{cases} \quad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
4. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\,\text{sech}^{-1}\frac{|x|}{a} + C;\ 0 < |x| < a \quad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + C$
5. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\,\text{csch}^{-1}\frac{|x|}{a} + C;\ x \neq 0,\ a > 0 \quad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right| + C$

We practice using the derivative and integral formulas in the following example.

Example 7.4.3 Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\dfrac{d}{dx}\left[\cosh^{-1}\left(\dfrac{3x-2}{5}\right)\right]$
2. $\displaystyle\int \frac{1}{x^2-1}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Notes:

Solution

1. Applying Key Idea 7.4.3 with the Chain Rule gives:
$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2 - 1}} \cdot \frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives a second integral can be solved with a direct application of item #3 from Key Idea 7.4.4, with $a = 1$. Thus
$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx \\ &= \begin{cases} -\tanh^{-1}(x) + C & x^2 < 1 \\ -\coth^{-1}(x) + C & 1 < x^2 \end{cases} \\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C. \qquad (7.4.1)\end{aligned}$$

3. This requires a substitution, then item #2 of Key Idea 7.4.4 can be applied. Let $u = 3x$, hence $du = 3\,dx$. We have
$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$
Note $a^2 = 10$, hence $a = \sqrt{10}$. Now apply the integral rule.
$$\begin{aligned} &= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right) + C \\ &= \frac{1}{3}\ln\left|3x + \sqrt{9x^2+10}\right| + C.\end{aligned}$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 7.4.4 contains perhaps the most useful information. Know the integration forms it helps

Notes:

evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer.

The next section takes a brief break from demonstrating new integration techniques. It instead demonstrates a technique of evaluating limits that return indeterminate forms. This technique will be useful in Section 8.6, where limits will arise in the evaluation of certain definite integrals.

Notes:

Exercises 7.4

Terms and Concepts

1. In Key Idea 7.4.1, the equation $\int \tanh x\,dx = \ln(\cosh x) + C$ is given. Why is "$\ln|\cosh x|$" not used — i.e., why are absolute values not necessary?
2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 7.4.1. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

3. Suppose $\sinh t = 5/12$. Find the values of the other five hyperbolic functions at t.
4. Suppose $\tanh t = -3/5$. Find the values of the other five hyperbolic functions at t.

In Exercises 5–12, verify the given identity using Definition 7.4.1, as done in Example 7.4.1.

5. $\coth^2 x - \operatorname{csch}^2 x = 1$
6. $\cosh 2x = \cosh^2 x + \sinh^2 x$
7. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
8. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$
9. $\dfrac{d}{dx}\left[\operatorname{sech} x\right] = -\operatorname{sech} x \tanh x$
10. $\dfrac{d}{dx}\left[\coth x\right] = -\operatorname{csch}^2 x$
11. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$
12. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

In Exercises 13–24, find the derivative of the given function.

13. $f(x) = \sinh 2x$
14. $f(x) = \cosh^2 x$
15. $f(x) = \tanh(x^2)$
16. $f(x) = \ln(\sinh x)$
17. $f(x) = \sinh x \cosh x$
18. $f(x) = x\sinh x - \cosh x$
19. $f(x) = \operatorname{sech}^{-1}(x^2)$
20. $f(x) = \sinh^{-1}(3x)$
21. $f(x) = \cosh^{-1}(2x^2)$
22. $f(x) = \tanh^{-1}(x+5)$
23. $f(x) = \tanh^{-1}(\cos x)$
24. $f(x) = \cosh^{-1}(\sec x)$

In Exercises 25–30, find the equation of the line tangent to the function at the given x-value.

25. $f(x) = \sinh x$ at $x = 0$
26. $f(x) = \cosh x$ at $x = \ln 2$
27. $f(x) = \tanh x$ at $x = -\ln 3$
28. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$
29. $f(x) = \sinh^{-1} x$ at $x = 0$
30. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

In Exercises 31–38, evaluate the given indefinite integral.

31. $\displaystyle\int \tanh(2x)\,dx$
32. $\displaystyle\int \cosh(3x-7)\,dx$
33. $\displaystyle\int \sinh x \cosh x\,dx$
34. $\displaystyle\int \frac{1}{9-x^2}\,dx$
35. $\displaystyle\int \frac{2x}{\sqrt{x^4-4}}\,dx$
36. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx$
37. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$
38. $\displaystyle\int \operatorname{sech} x\,dx$ (Hint: multiply by $\frac{\cosh x}{\cosh x}$; set $u = \sinh x$.)

In Exercises 39–40, evaluate the given definite integral.

39. $\displaystyle\int_{-1}^{1} \sinh x\,dx$
40. $\displaystyle\int_{-\ln 2}^{\ln 2} \cosh x\,dx$
41. In the bottom graph of Figure 7.4.1 (the hyperbola), it is stated that the shaded area is $\theta/2$. Verify this claim by setting up and evaluating an appropriate integral (and note that θ is just a positive number, not an angle).
Hint: Integrate with respect to y, and consult the table of Integration Rules in the Appendix if necessary.

7.5 L'Hôpital's Rule

This section is concerned with a technique for evaluating certain limits that will be useful in later chapters.

Our treatment of limits exposed us to "0/0", an indeterminate form. If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ are zero, we do not conclude that $\lim_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces L'Hôpital's Rule, a method of resolving limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem 7.5.1 L'Hôpital's Rule, Part 1
Let f and g be differentiable functions on an open interval I containing a.

1. If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $g'(x) \neq 0$ except possibly at $x = a$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
assuming that the limit on the right exists.

2. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
assuming that the limit on the right exists.

Notes:

A similar statement holds if we just look at the one sided limits $\lim\limits_{x\to a^-}$ and $\lim\limits_{x\to a^+}$.

Theorem 7.5.2 L'Hôpital's Rule, Part 2

Let f and g be differentiable functions on the open interval (c, ∞) for some value c and $g'(x) \neq 0$ on (c, ∞).

1. If $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} g(x) = 0$, then
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$
assuming that the limit on the right exists.

2. If $\lim\limits_{x\to\infty} f(x) = \pm\infty$ and $\lim\limits_{x\to\infty} g(x) = \pm\infty$, then
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$
assuming that the limit on the right exists.

Similar statements can be made where x approaches $-\infty$.

We demonstrate the use of L'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "L'Hôpital's Rule."

Example 7.5.1 Using L'Hôpital's Rule

Evaluate the following limits, using L'Hôpital's Rule as needed.

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$
2. $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{1-x}$
3. $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x}$
4. $\lim\limits_{x\to -3} \dfrac{x^3+27}{x^2+9}$
5. $\lim\limits_{x\to\infty} \dfrac{3x^2-100x+2}{4x^2+5x-1000}$
6. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^3}$

Solution

1. This has the indeterminate form $0/0$. We proved this limit is 1 in Example 1.3.4 using the Squeeze Theorem. Here we use L'Hôpital's Rule to show

Notes:

its power.

$$\lim_{x\to 0}\frac{\sin x}{x} \overset{\text{by LHR}}{=} \lim_{x\to 0}\frac{\cos x}{1} = 1.$$

While this seems easier than using the Squeeze Theorem to find this limit, we note that applying L'Hôpital's Rule here requires us to know the derivative of $\sin x$. We originally encountered this limit when we were trying to find that derivative.

2. This has the indeterminate form $0/0$.

$$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{1-x} \overset{\text{by LHR}}{=} \lim_{x\to 1}\frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$

3. This has the indeterminate form $0/0$.

$$\lim_{x\to 0}\frac{x^2}{1-\cos x} \overset{\text{by LHR}}{=} \lim_{x\to 0}\frac{2x}{\sin x}.$$

This latter limit also evaluates to the $0/0$ indeterminate form. To evaluate it, we apply L'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{2x}{\sin x} \overset{\text{by LHR}}{=} \lim_{x\to 0}\frac{2}{\cos x} = 2.$$

Thus $\displaystyle\lim_{x\to 0}\frac{x^2}{1-\cos x} = 2.$

4. $$\lim_{x\to -3}\frac{x^3+27}{x^2+9} = \frac{0}{18} = 0$$

We cannot use L'Hôpital's Rule in this case because the original limit does not return an indeterminate form, so L'Hôpital's Rule does not apply. In fact, the inappropriate use of L'Hôpital's Rule here would result in the incorrect limit $-\frac{9}{2}$.

5. We can evaluate this limit already using Key Idea 1.5.2; the answer is 3/4. We apply L'Hôpital's Rule to demonstrate its applicability.

$$\lim_{x\to\infty}\frac{3x^2-100x+2}{4x^2+5x-1000} \overset{\text{by LHR}}{=} \lim_{x\to\infty}\frac{6x-100}{8x+5} \overset{\text{by LHR}}{=} \lim_{x\to\infty}\frac{6}{8} = \frac{3}{4}.$$

6. $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^3} \overset{\text{by LHR}}{=} \lim_{x\to\infty}\frac{e^x}{3x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty}\frac{e^x}{6x} \overset{\text{by LHR}}{=} \lim_{x\to\infty}\frac{e^x}{6} = \infty.$

Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This has important implications in computing when considering efficiency of algorithms.)

Notes:

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that L'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Watch the video:
L'Hôpital's Rule — Indeterminate Powers at
`https://youtu.be/kEnwac_9lyg`

Example 7.5.2 Applying L'Hôpital's Rule to other indeterminate forms

Evaluate the following limits.

1. $\lim\limits_{x\to 0^+} x \cdot e^{1/x}$
2. $\lim\limits_{x\to 0^-} x \cdot e^{1/x}$
3. $\lim\limits_{x\to \infty} (\ln(x+1) - \ln x)$
4. $\lim\limits_{x\to \infty} \left(x^2 - e^x\right)$

Solution

1. As $x \to 0^+$, note that $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which L'Hôpital's Rule can be applied.

$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \overset{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$

Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, note that $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that

$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

Notes:

3. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as

$$\lim_{x\to\infty} \left(\ln(x+1) - \ln x\right) = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$

As $x \to \infty$, the argument of the natural logarithm approaches ∞/∞, to which we can apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x+1}{x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{1}{1} = 1.$$

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \left(\ln(x+1) - \ln x\right) = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\lim\limits_{x\to\infty} \left(x^2 - e^x\right)$ initially returns the indeterminate form $\infty - \infty$.

We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$.

We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} \left(x^2 - e^x\right) = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Notes:

Indeterminate Forms 0^0, 1^∞, and ∞^0

When faced with a limit that returns one of the indeterminate forms 0^0, 1^∞, or ∞^0, it is often useful to use the natural logarithm to convert to an indeterminate form we already know how to find the limit of, then use the natural exponential function find the original limit. This is possible because the natural logarithm and natural exponential functions are inverses and because they are both continuous. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 7.5.1 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\displaystyle\lim_{x\to c} \ln\big(f(x)\big) = L$, then $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} e^{\ln(f(x))} = e^{L}$.

Example 7.5.3 Using L'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

$$1.\quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \qquad\qquad 2.\quad \lim_{x\to 0^+} x^x.$$

Solution

1. This is equivalent to a special limit given in Theorem 1.3.6; these limits have important applications in mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\displaystyle\lim_{x\to\infty} f(x)$. Let's first evaluate $\displaystyle\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty} \ln\big(f(x)\big) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x \\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

Notes:

This produces the indeterminate form 0/0, so we apply L'Hôpital's Rule.

$$\begin{aligned} &= \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot\left(-1/x^2\right)}{\left(-1/x^2\right)} \\ &= \lim_{x\to\infty} \frac{1}{1+1/x} \\ &= 1. \end{aligned}$$

Thus $\lim\limits_{x\to\infty} \ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 7.5.1.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

This is another way to determine the value of the number e.

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider first $\lim\limits_{x\to 0^+} \ln\big(f(x)\big)$.

$$\begin{aligned} \lim_{x\to 0^+} \ln\big(f(x)\big) &= \lim_{x\to 0^+} \ln\left(x^x\right) \\ &= \lim_{x\to 0^+} x\ln x \end{aligned}$$

This produces the indeterminate form $0(-\infty)$, so we rewrite it in order to apply L'Hôpital's Rule.

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

This produces the indeterminate form $-\infty/\infty$ so we apply L'Hôpital's Rule.

$$\begin{aligned} &= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0^+} -x \\ &= 0. \end{aligned}$$

Thus $\lim\limits_{x\to 0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 7.5.1.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 7.5.1.

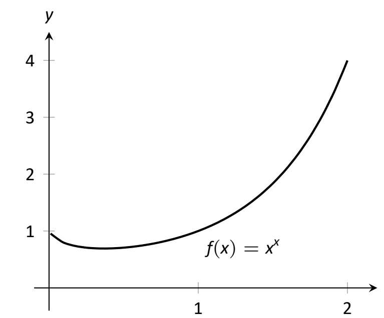


Figure 7.5.1: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+$, $f(x) \to 1$.

Notes:

Exercises 7.5

Terms and Concepts

1. List the different indeterminate forms described in this section.
2. T/F: L'Hôpital's Rule provides a faster method of computing derivatives.
3. T/F: L'Hôpital's Rule states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$.
4. Explain what the indeterminate form "1^∞" means.
5. Fill in the blanks: The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking ________; L'Hôpital's Rule is applied when taking certain ________.
6. Create (but do not evaluate!) a limit that returns "∞^0".
7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".
8. Create a function $f(x)$ such that $\lim\limits_{x\to\infty} f(x)$ returns "$0 \cdot \infty$".

Problems

In Exercises 9–54, evaluate the given limit.

9. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1}$
10. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$
11. $\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi}$
12. $\lim\limits_{x\to \pi/4} \frac{\sin x-\cos x}{\cos(2x)}$
13. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x}$
14. $\lim\limits_{x\to 0} \frac{\sin(2x)}{x+2}$
15. $\lim\limits_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$
16. $\lim\limits_{x\to 0^+} \frac{e^x-1}{x^2}$
17. $\lim\limits_{x\to 0^+} \frac{e^x-x-1}{x^2}$
18. $\lim\limits_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$
19. $\lim\limits_{x\to \infty} \frac{x^4}{e^x}$
20. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{e^x}$
21. $\lim\limits_{x\to \infty} \frac{e^x}{\sqrt{x}}$
22. $\lim\limits_{x\to \infty} \frac{e^x}{2^x}$
23. $\lim\limits_{x\to \infty} \frac{e^x}{3^x}$
24. $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$
25. $\lim\limits_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$
26. $\lim\limits_{x\to \infty} \frac{\ln x}{x}$
27. $\lim\limits_{x\to \infty} \frac{\ln(x^2)}{x}$
28. $\lim\limits_{x\to \infty} \frac{\left(\ln x\right)^2}{x}$
29. $\lim\limits_{x\to 0^+} x\ln x$
30. $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$
31. $\lim\limits_{x\to 0^+} xe^{1/x}$
32. $\lim\limits_{x\to \infty} \left(x^3-x^2\right)$
33. $\lim\limits_{x\to \infty} \left(\sqrt{x}-\ln x\right)$
34. $\lim\limits_{x\to -\infty} xe^x$
35. $\lim\limits_{x\to 0^+} \frac{1}{x^2}e^{-1/x}$
36. $\lim\limits_{x\to 0^+} (1+x)^{1/x}$
37. $\lim\limits_{x\to 0^+} (2x)^x$
38. $\lim\limits_{x\to 0^+} (2/x)^x$
39. $\lim\limits_{x\to 0^+} (\sin x)^x$
40. $\lim\limits_{x\to 1^-} (1-x)^{1-x}$
41. $\lim\limits_{x\to \infty} (x)^{1/x}$
42. $\lim\limits_{x\to \infty} (1/x)^x$
43. $\lim\limits_{x\to 1^+} (\ln x)^{1-x}$
44. $\lim\limits_{x\to \infty} (1+x)^{1/x}$
45. $\lim\limits_{x\to \infty} (1+x^2)^{1/x}$
46. $\lim\limits_{x\to \pi/2} \tan x\cos x$
47. $\lim\limits_{x\to \pi/2} \tan x\sin(2x)$
48. $\lim\limits_{x\to 1^+} \left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$
49. $\lim\limits_{x\to 3^+} \left(\frac{5}{x^2-9}-\frac{x}{x-3}\right)$
50. $\lim\limits_{x\to \infty} x\tan(1/x)$
51. $\lim\limits_{x\to \infty} \frac{(\ln x)^3}{x}$
52. $\lim\limits_{x\to 1} \frac{x^2+x-2}{\ln x}$
53. $\lim\limits_{x\to 0^+} \left(\frac{2+5^x}{3}\right)^{1/x}$
54. $\lim\limits_{x\to \infty} \left(\frac{1+7^{1/x}}{2}\right)^x$
55. Following the guidelines in Section 3.5, and using L'Hôpital's rule where appropriate, neatly sketch the graph of $f(x) = \frac{\ln(x)}{\ln(2x)}$. Check your answer using a graphing utility (and be careful near 0).

8: Techniques of Integration

Chapter 5 introduced the antiderivative and connected it to signed areas under a curve through the Fundamental Theorem of Calculus. The chapter after explored more applications of definite integrals than just area. As evaluating definite integrals will become even more important, we will want to find antiderivatives of a variety of functions.

This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions, we can still find antiderivatives of a wide variety of functions.

8.1 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x\cos x\,dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v+uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)'\,dx = \int (u'v + uv')\,dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v\,dx + \int uv'\,dx.$$

Notes:

Solving for the second integral we have

$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x.$$

Using differential notation, we can write

$$\begin{aligned} u' &= \frac{\mathrm{d}u}{\mathrm{d}x} \\ v' &= \frac{\mathrm{d}v}{\mathrm{d}x} \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \mathrm{d}u &= u'\,\mathrm{d}x \\ \mathrm{d}v &= v'\,\mathrm{d}x. \end{aligned}$$

Thus, the equation above can be written as follows:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 8.1.1 Integration by Parts
Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u,$$

and applying FTC part 2 we have

$$\int_{x=a}^{x=b} u\,\mathrm{d}v = uv\Big|_a^b - \int_{x=a}^{x=b} v\,\mathrm{d}u.$$

Watch the video:
Integration by Parts — Definite Integral at
`https://youtu.be/zGGI4PkHzhI`

Let's try an example to understand our new technique.

Notes:

Example 8.1.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x\cos x\,\mathrm{d}x$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $\mathrm{d}v = \cos x\,\mathrm{d}x$.

It is generally useful to make a small table of these values.

$$\begin{array}{ll} u = x & \mathrm{d}v = \cos x\,\mathrm{d}x \\ \mathrm{d}u = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = x & \mathrm{d}v = \cos x\,\mathrm{d}x \\ \mathrm{d}u = \mathrm{d}x & v = \sin x \end{array}$$

Right now we only know u and dv as shown on the left; on the right we fill in the rest of what we need. If $u = x$, then $\mathrm{d}u = \mathrm{d}x$. Since $\mathrm{d}v = \cos x\,\mathrm{d}x$, v is an antiderivative of $\cos x$, so $v = \sin x$.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x\cos x\,\mathrm{d}x = x\sin x + \cos x + C.$$

We have two important notes here: (1) notice how the antiderivative contains the product, $x\sin x$. This product is what makes integration by parts necessary. And (2) antidifferentiating dv does result in $v + C$. The intermediate $+C$s are all added together and represented by one $+C$ in the final answer.

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,\mathrm{d}u$ will be simpler to integrate than the original integral $\int u\,\mathrm{d}v$.

In the example above, we chose $u = x$ and $\mathrm{d}v = \cos x\,\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x\cos x\,\mathrm{d}x$, we could integrate $\sin x\,\mathrm{d}x$, which we knew how to do.

If we had chosen $u = \cos x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = -\sin x\,\mathrm{d}x$ and $v = \frac{1}{2}x^2$, then

$$\int x\cos x\,\mathrm{d}x = \frac{1}{2}x^2\cos x - \left(-\frac{1}{2}\right)\int x^2\sin x\,\mathrm{d}x.$$

Notes:

We then need to integrate $x^2 \sin x$, which is more complicated than our original integral, making this an unproductive choice.

We now consider another example.

Example 8.1.2 Integrating using Integration by Parts

Evaluate $\displaystyle\int xe^x \, dx$.

Solution Notice that x becomes simpler when differentiated and e^x is unchanged by differentiation or integration. This suggests that we should let $u = x$ and $dv = e^x \, dx$:

$$\begin{array}{cc} u = x & dv = e^x \, dx \\ du = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & dv = e^x \, dx \\ du = dx & v = e^x \end{array}$$

The Integration by Parts formula gives

$$\int xe^x \, dx = xe^x - \int e^x \, dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x \, dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 8.1.3 Integrating using Integration by Parts

Evaluate $\displaystyle\int x^2 \cos x \, dx$.

Solution Let $u = x^2$ instead of the trigonometric function, hence $dv = \cos x \, dx$. Then $du = 2x \, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & dv = \cos x \, dx \\ du = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & dv = \cos x \, dx \\ du = 2x \, dx & v = \sin x \end{array}$$

The Integration by Parts formula gives

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we

Notes:

choose $u = 2x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & \mathrm{d}v = \sin x\,\mathrm{d}x \\ \mathrm{d}u = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & \mathrm{d}v = \sin x\,\mathrm{d}x \\ \mathrm{d}u = 2\,\mathrm{d}x & v = -\cos x \end{array}$$

This means that

$$\int x^2 \cos x\,\mathrm{d}x = x^2 \sin x - \left(-2x\cos x - \int -2\cos x\,\mathrm{d}x\right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2\sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x\,\mathrm{d}x = x^2 \sin x + 2x\cos x - 2\sin x + C.$$

Example 8.1.4 Integrating using Integration by Parts

Evaluate $\displaystyle\int e^x \cos x\,\mathrm{d}x$.

Solution This is a classic problem. In this particular example, one can let u be either $\cos x$ or e^x; we choose $u = e^x$ and hence $\mathrm{d}v = \cos x\,\mathrm{d}x$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = e^x & \mathrm{d}v = \cos x\,\mathrm{d}x \\ \mathrm{d}u = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & \mathrm{d}v = \cos x\,\mathrm{d}x \\ \mathrm{d}u = e^x\,\mathrm{d}x & v = \sin x \end{array}$$

Notice that $\mathrm{d}u$ is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\,\mathrm{d}x = e^x \sin x - \int e^x \sin x\,\mathrm{d}x.$$

The integral on the right is not much different from the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral. So what should we use for u and $\mathrm{d}v$ this time? We may feel like letting the trigonometric function be $\mathrm{d}v$ and the exponential be u was a bad choice last time since we still can't integrate the new integral. However, if we let $u = \sin x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$ this time we will reverse what we just did, taking us back to the beginning. So, we let $u = e^x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$. This leads us to the following:

$$\begin{array}{cc} u = e^x & \mathrm{d}v = \sin x\,\mathrm{d}x \\ \mathrm{d}u = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & \mathrm{d}v = \sin x\,\mathrm{d}x \\ \mathrm{d}u = e^x\,\mathrm{d}x & v = -\cos x \end{array}$$

Notes:

The Integration by Parts formula then gives:

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x \, dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x \, dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x \, dx$ to both sides. This gives

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x \, dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x \, dx = \frac{1}{2} e^x \left(\sin x + \cos x\right) + C.$$

Example 8.1.5 Using Integration by Parts: antiderivative of $\ln x$

Evaluate $\displaystyle\int \ln x \, dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good strategy to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{cc} u = \ln x & dv = dx \\ du = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = \ln x & dv = dx \\ du = 1/x \, dx & v = x \end{array}$$

Putting this all together in the Integration by Parts formula, things work out very

Notes:

nicely:

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C.\end{aligned}$$

Example 8.1.6 Using Integration by Parts: antiderivative of $\tan^{-1} x$

Evaluate $\displaystyle\int \tan^{-1} x \, dx$.

Solution The same strategy of $dv = dx$ that we used above works here. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = 1/(1 + x^2)\, dx$ and $v = x$. The Integration by Parts formula gives

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1 + x^2} \, dx.$$

The integral on the right can be solved by substitution. Taking $t = 1 + x^2$, we get $dt = 2x \, dx$. The integral then becomes

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{t} \, dt.$$

The integral on the right evaluates to $\ln|t| + C$, which becomes $\ln(1 + x^2) + C$. Therefore, the answer is

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C.$$

Since $1 + x^2 > 0$, we do not need to include the absolute value in the $\ln(1 + x^2)$ term.

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Notes:

Example 8.1.7 Integration by Parts after substitution

Evaluate $\displaystyle\int \cos(\ln x)\,dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\,dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\,dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions to solve for $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\,du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\,du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\,dx = \int e^u \cos u\,du.$$

We evaluated this integral in Example 8.1.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\,dx &= \int e^u \cos u\,du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 8.1.1 states. We do so in the next example.

Example 8.1.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\,dx$.

Solution To simplify the integral we let $u = \ln x$ and $dv = x^2\,dx$. We

Notes:

then get $\text{d}u = (1/x)\,\text{d}x$ and $v = x^3/3$ as shown below.

$$\begin{array}{cc} u = \ln x & \text{d}v = x^2\,\text{d}x \\ \text{d}u = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = \ln x & \text{d}v = x^2\,\text{d}x \\ \text{d}u = 1/x\,\text{d}x & v = x^3/3 \end{array}$$

This may seem counterintuitive since the power on the algebraic factor has increased ($v = x^3/3$), but as we see this is a wise choice:

$$\begin{aligned}
\int_1^2 x^2 \ln x\,\text{d}x &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\,\frac{1}{x}\,\text{d}x \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\,\text{d}x \\
&= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\
&= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\
&= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\
&= \frac{8}{3}\ln 2 - \frac{7}{9}.
\end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\,\text{d}x$ or $\int x^3 \sin x\,\text{d}x$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than differentiation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar-looking integrals

$$\int xe^x\,\text{d}x, \qquad \int xe^{x^2}\,\text{d}x \qquad \text{and} \qquad \int xe^{x^3}\,\text{d}x.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer. We will learn how to approximate this integral in Chapter 9

Integration by Parts is a very useful method, second only to substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Notes:

Exercises 8.1

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.
2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

Problems

In Exercises 3–36, evaluate the given indefinite integral.

3. $\int x \sin x \, dx$
4. $\int x e^{-x} \, dx$
5. $\int x^2 \sin x \, dx$
6. $\int x^3 \sin x \, dx$
7. $\int x e^{x^2} \, dx$
8. $\int x^3 e^x \, dx$
9. $\int x e^{-2x} \, dx$
10. $\int e^x \sin x \, dx$
11. $\int e^{2x} \cos x \, dx$
12. $\int e^{2x} \sin(3x) \, dx$
13. $\int e^{5x} \cos(5x) \, dx$
14. $\int \sin x \cos x \, dx$
15. $\int \sin^{-1} x \, dx$
16. $\int \tan^{-1}(2x) \, dx$
17. $\int x \tan^{-1} x \, dx$
18. $\int \cos^{-1} x \, dx$
19. $\int x \ln x \, dx$
20. $\int (x-2) \ln x \, dx$
21. $\int x \ln(x-1) \, dx$
22. $\int x \ln(x^2) \, dx$
23. $\int x^2 \ln x \, dx$
24. $\int (\ln x)^2 \, dx$
25. $\int (\ln(x+1))^2 \, dx$
26. $\int x \sec^2 x \, dx$
27. $\int x \csc^2 x \, dx$
28. $\int x\sqrt{x-2} \, dx$
29. $\int x\sqrt{x^2-2} \, dx$
30. $\int \sec x \tan x \, dx$
31. $\int x \sec x \tan x \, dx$
32. $\int x \csc x \cot x \, dx$
33. $\int x \cosh x \, dx$
34. $\int x \sinh x \, dx$
35. $\int \sinh^{-1} x \, dx$
36. $\int \tanh^{-1} x \, dx$

In Exercises 37–42, evaluate the indefinite integral after first making a substitution.

37. $\int \sin(\ln x) \, dx$
38. $\int \sin(\sqrt{x}) \, dx$
39. $\int \ln(\sqrt{x}) \, dx$
40. $\int e^{\sqrt{x}} \, dx$
41. $\int e^{\ln x} \, dx$
42. $\int x^3 e^{x^2} \, dx$

In Exercises 43–52, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 3–12.

43. $\int_0^{\pi} x \sin x \, dx$
44. $\int_{-1}^{1} x e^{-x} \, dx$
45. $\int_{-\pi/4}^{\pi/4} x^2 \sin x \, dx$
46. $\int_{-\pi/2}^{\pi/2} x^3 \sin x \, dx$
47. $\int_0^{\sqrt{\ln 2}} x e^{x^2} \, dx$

48. $\int_0^1 x^3 e^x \, dx$

49. $\int_1^2 x e^{-2x} \, dx$

50. $\int_0^\pi e^x \sin x \, dx$

51. $\int_{-\pi/2}^{\pi/2} e^{2x} \cos x \, dx$

52. $\int_0^{\pi/3} e^{2x} \sin(3x) \, dx$

53. (a) For $n \geq 2$ show that

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx.$$

Hint: Begin by writing $\sin^n x$ as $(\sin^{n-1} x) \sin x$ and using Integration by Parts.

(b) For $k \geq 1$ show that

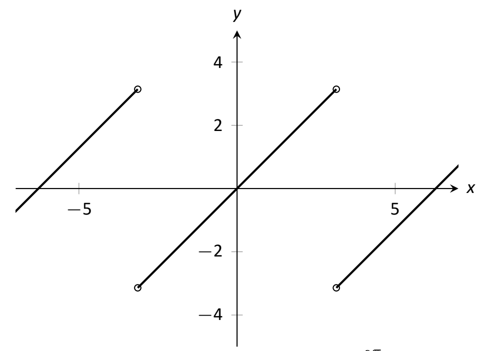
$$\int_0^{\pi/2} \sin^{2k} x \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)} \frac{\pi}{2} \quad \text{and}$$
$$\int_0^{\pi/2} \sin^{2k+1} x \, dx = \frac{2 \cdot 4 \cdot 6 \cdots (2k)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)}.$$

54. Find the volume of the solid of revolution obtained by rotating the region bounded by $y = 0$, $y = \ln x$, $x = 1$, and $x = e$:

(a) About the x-axis, using the disk method.

(b) About the y-axis, using the shell method.

55. Let $f(x) = x$ for $-\pi \leq x < \pi$ and extend this function so that it is periodic with period 2π. This function is known as a *sawtooth wave* and looks like

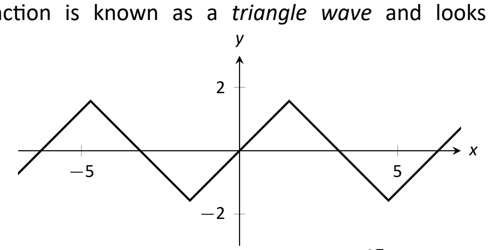


For a positive integer n, define $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$.

(a) Find b_n.

(b) Graph $\sum_{n=1}^{N} b_n \sin(nx)$ for various values of N. What do you observe?

56. Let $f(x) = \begin{cases} -x - \pi & -\pi \leq x < -\frac{\pi}{2} \\ x & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x < \pi \end{cases}$ and extend this function so that it is periodic with period 2π. This function is known as a *triangle wave* and looks like



For a positive integer n, define $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$.

(a) Find b_n.

(b) Graph $\sum_{n=1}^{N} b_n \sin(nx)$ for various values of N. What do you observe?

8.2 Trigonometric Integrals

Trigonometric functions are useful for describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\displaystyle\int \sin^m x \cos^n x \,dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x \,dx$ in Example 5.5.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this and consider integrals of the form $\int \sin^m x \cos^n x \,dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Watch the video:
Trigonometric Integrals — Part 2 of 6 at
`https://youtu.be/zyg9k1je7Fg`

Key Idea 8.2.1 Integrals Involving Powers of Sine and Cosine

Consider $\displaystyle\int \sin^m x \cos^n x \,dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite

$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$

   Then

$$\int \sin^m x \cos^n x \,dx = \int (1 - \cos^2 x)^k \sin x \cos^n x \,dx = -\int (1 - u^2)^k u^n \,du,$$

   where $u = \cos x$ and $du = -\sin x \,dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

$$\int \sin^m x \cos^n x \,dx = \int u^m (1 - u^2)^k \,du,$$

(continued)

Notes:

Key Idea 8.2.1 continued

where $u = \sin x$ and $\mathrm{d}u = \cos x\,\mathrm{d}x$.

3. If both m and n are even, use the half-angle identities

$$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

We practice applying Key Idea 8.2.1 in the next examples.

Example 8.2.1 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x \cos^8 x\,\mathrm{d}x$.

Solution The power of the sine factor is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

Our integral is now $\displaystyle\int (1-\cos^2 x)^2 \cos^8 x \sin x\,\mathrm{d}x$. Let $u = \cos x$, hence $\mathrm{d}u = -\sin x\,\mathrm{d}x$. Making the substitution and expanding the integrand gives

$$\begin{aligned}
\int (1-\cos^2 x)^2 \cos^8 x \sin x\,\mathrm{d}x &= -\int (1-u^2)^2 u^8\,\mathrm{d}u \\
&= -\int \left(1-2u^2+u^4\right)u^8\,\mathrm{d}u \\
&= -\int \left(u^8-2u^{10}+u^{12}\right)\mathrm{d}u \\
&= -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C \\
&= -\frac{1}{9}\cos^9 x+\frac{2}{11}\cos^{11} x-\frac{1}{13}\cos^{13} x+C.
\end{aligned}$$

Example 8.2.2 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x \cos^9 x\,\mathrm{d}x$.

Solution Because the powers of both the sine and cosine factors are odd, we can apply the techniques of Key Idea 8.2.1 to either power. We choose to work with the power of the sine factor since that has a smaller exponent.

Notes:

We rewrite $\sin^5 x$ as

$$\begin{aligned}\sin^5 x &= \sin^4 x \sin x\\ &= (1-\cos^2 x)^2 \sin x\\ &= (1-2\cos^2 x+\cos^4 x)\sin x.\end{aligned}$$

This lets us rewrite the integral as

$$\int \sin^5 x \cos^9 x \,dx = \int \big(1-2\cos^2 x+\cos^4 x\big)\sin x\cos^9 x\,dx.$$

Substituting and integrating with $u=\cos x$ and $du=-\sin x\,dx$, we have

$$\begin{aligned}&\int \big(1-2\cos^2 x+\cos^4 x\big)\sin x\cos^9 x\,dx\\ &\qquad= -\int \big(1-2u^2+u^4\big)u^9\,du\\ &\qquad= -\int u^9-2u^{11}+u^{13}\,du\\ &\qquad= -\frac{1}{10}u^{10}+\frac{1}{6}u^{12}-\frac{1}{14}u^{14}+C\\ &\qquad= -\frac{1}{10}\cos^{10}x+\frac{1}{6}\cos^{12}x-\frac{1}{14}\cos^{14}x+C.\end{aligned}$$

Instead, another approach would be to rewrite $\cos^9 x$ as

$$\begin{aligned}\cos^9 x &= \cos^8 x\cos x\\ &= (\cos^2 x)^4\cos x\\ &= (1-\sin^2 x)^4\cos x\\ &= (1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x)\cos x.\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x\cos^9 x\,dx = \int \big(\sin^5 x\big)\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\,dx.$$

Notes:

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\, dx$.

$$\begin{aligned}
&\int \left(\sin^5 x\right)\left(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\right)\cos x\, dx \\
&\quad = \int u^5(1 - 4u^2 + 6u^4 - 4u^6 + u^8)\, du \\
&\quad = \int \left(u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13}\right) du \\
&\quad = \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C \\
&\quad = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.
\end{aligned}$$

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x \cos^9 x\, dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our second answer in Example 8.2.2, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 8.2.1 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*: $g(x) = f(x) + C$ for some constant C. We have two different antiderivatives of the same function, meaning both answers are correct.

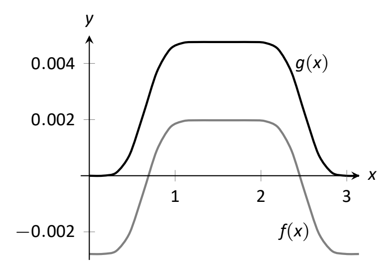


Figure 8.2.1: A plot of $f(x)$ and $g(x)$ from Example 8.2.2 and the Technology Note.

Example 8.2.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^2 x\, dx$.

SOLUTION The power of sine is even so we employ a half-angle identity,

Notes:

algebra and a u- substitution as follows:

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos(2x)}{2}\,dx\\ &= \frac{1}{2}\int 1-\cos(2x)\,dx\\ &= \frac{1}{2}\left(x-\frac{1}{2}\sin(2x)\right)+C\\ &= \frac{1}{2}x-\frac{1}{4}\sin(2x)+C.\end{aligned}$$

Example 8.2.4 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x\sin^2 x\,dx$.

Solution The powers of sine and cosine are both even, so we employ the half-angle formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)dx\\ &= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\,dx\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the half-angle formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x)+C.$$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big)+C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Notes:

Letting $u = \sin(2x)$, we have $\mathrm{d}u = 2\cos(2x)\,\mathrm{d}x$, hence

$$\begin{aligned}\int \cos^3(2x)\,\mathrm{d}x &= \int \big(1-\sin^2(2x)\big)\cos(2x)\,\mathrm{d}x\\ &= \int \frac{1}{2}(1-u^2)\,\mathrm{d}u\\ &= \frac{1}{2}\Big(u-\frac{1}{3}u^3\Big)+C\\ &= \frac{1}{2}\Big(\sin(2x)-\frac{1}{3}\sin^3(2x)\Big)+C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}&\int \cos^4 x\sin^2 x\,\mathrm{d}x\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,\mathrm{d}x\\ &= \frac{1}{8}\Big[x+\frac{1}{2}\sin(2x)-\frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big)-\frac{1}{2}\Big(\sin(2x)-\frac{1}{3}\sin^3(2x)\Big)\Big]+C\\ &= \frac{1}{8}\Big[\frac{1}{2}x-\frac{1}{8}\sin(4x)+\frac{1}{6}\sin^3(2x)\Big]+C.\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\displaystyle\int \tan^m x\sec^n x\,\mathrm{d}x$

When evaluating integrals of the form $\int \sin^m x\cos^n x\,\mathrm{d}x$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vice versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x\sec^n x\,\mathrm{d}x$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$,
- $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x\tan x$, and
- $1+\tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

Notes:

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x \tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 8.2.2 Integrals Involving Powers of Tangent and Secant

Consider $\displaystyle\int \tan^m x \sec^n x\,dx$, where m and n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1+\tan^2 x)^{k-1}\sec^2 x.$$

   Then

$$\int \tan^m x \sec^n x\,dx = \int \tan^m x(1+\tan^2 x)^{k-1}\sec^2 x\,dx = \int u^m(1+u^2)^{k-1}\,du,$$

   where $u = \tan x$ and $du = \sec^2 x\,dx$.

2. If m is odd and $n > 0$, then $m = 2k+1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

$$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$

   Then

$$\int \tan^m x \sec^n x\,dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\,dx = \int (u^2-1)^k u^{n-1}\,du,$$

   where $u = \sec x$ and $du = \sec x \tan x\,dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\,dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\sec^2 x - 1) = \tan^{m-2}\sec^2 x - \tan^{m-2} x.$$

   So

$$\int \tan^m x\,dx = \underbrace{\int \tan^{m-2} x \sec^2 x\,dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\,dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 of Key Idea 8.2.2 are relatively

Notes:

straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Example 8.2.5 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^2 x \sec^6 x \, dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 8.2.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x \, dx &= \int \tan^2 x \sec^4 x \sec^2 x \, dx\\ &= \int \tan^2 x \big(1+\tan^2 x\big)^2 \sec^2 x \, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x \, dx$.

$$= \int u^2 \big(1+u^2\big)^2 \, du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

We derived integrals for tangent and secant in Section 5.5 and will regularly use them when evaluating integrals of the form $\tan^m x \sec^n x \, dx$. As a reminder:

$$\begin{aligned}\int \tan x \, dx &= \ln|\sec x| + C\\ \int \sec x \, dx &= \ln|\sec x + \tan x| + C\end{aligned}$$

Example 8.2.6 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \sec^3 x \, dx$.

Solution We apply rule #3 from Key Idea 8.2.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x \, dx$, meaning that $u = \sec x$.

Notes:

$$\begin{array}{ll} u = \sec x & dv = \sec^2 x\,dx \\ du = ? & v = ? \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & dv = \sec^2 x\,dx \\ du = \sec x \tan x\,dx & v = \tan x \end{array}$$

Figure 8.2.2: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned} \int \sec^3 x\,dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x\,dx. \end{aligned}$$

This new integral also requires applying rule #3 of Key Idea 8.2.2:

$$\begin{aligned} &= \sec x \tan x - \int \sec x\big(\sec^2 x - 1\big)\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| \end{aligned}$$

Note: Remember that in Example 5.5.8, we found that $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\,dx$ to both sides, giving:

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| \\ \int \sec^3 x\,dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C. \end{aligned}$$

We give one more example.

Example 8.2.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x\,dx$.

Notes:

Solution We employ rule #4 of Key Idea 8.2.2.

$$\begin{aligned}\int \tan^6 x\,dx &= \int \tan^4 x \tan^2 x\,dx\\ &= \int \tan^4 x\big(\sec^2 x - 1\big)\,dx\\ &= \int \tan^4 x\sec^2 x\,dx - \int \tan^4 x\,dx\end{aligned}$$

We integrate the first integral with substitution, $u = \tan x$ and $du = \sec^2 x\,dx$; and the second by employing rule #4 again.

$$\begin{aligned}&= \int u^4\,du - \int \tan^2 x\tan^2 x\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x\sec^2 x\,dx + \int \tan^2 x\,dx\end{aligned}$$

Again, use substitution for the first integral and rule #4 for the second.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.\end{aligned}$$

Integrals of the form $\int \cot^m x \csc^n x\,dx$

Not surprisingly, evaluating integrals of the form $\int \cot^m x\csc^n x\,dx$ is similar to evaluating $\int \tan^m x\sec^n x\,dx$. The guidelines from Key Idea 8.2.2 and the following three facts will be useful:

$$\begin{aligned}\frac{d}{dx}(\cot x) &= -\csc^2 x\\ \frac{d}{dx}(\csc x) &= -\csc x\cot x, \qquad \text{and}\\ \csc^2 x &= \cot^2 x + 1\end{aligned}$$

Notes:

Example 8.2.8 Integrating powers of cotangent and cosecant

Evaluate $\displaystyle\int \cot^2 x \csc^4 x \, dx$

Solution Since the power of cosecant is even we will let $u = \cot x$ and save a $\csc^2 x$ for the resulting $du = -\csc^2 x \, dx$.

$$\begin{aligned}\int \cot^2 x \csc^4 x \, dx &= \int \cot^2 x \csc^2 x \csc^2 x \, dx \\ &= \int \cot^2 x(1+\cot^2 x)\csc^2 x \, dx \\ &= -\int u^2(1+u^2)\, du.\end{aligned}$$

The integration and substitution required to finish this example are similar to that of previous examples in this section. The result is

$$-\frac{1}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C.$$

Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\,dx$$

are best approached by first applying the Product to Sum Formulas of Trigonometry found in the back cover of this text, namely

$$\begin{aligned}\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big] \\ \cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big] \\ \sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]\end{aligned}$$

Notes:

Example 8.2.9 Integrating products of $\sin(mx)$ **and** $\cos(nx)$

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\,dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x)\,dx &= \int \frac{1}{2}\Big[\sin(3x)+\sin(7x)\Big]\,dx\\ &= -\frac{1}{6}\cos(3x)-\frac{1}{14}\cos(7x)+C.\end{aligned}$$

Integrating other combinations of trigonometric functions

Combinations of trigonometric functions that we have not discussed in this chapter are evaluated by applying algebra, trigonometric identities and other integration strategies to create an equivalent integrand that we can evaluate. To evaluate "crazy" combinations, those not readily manipulated into a familiar form, one should use integral tables. A table of "common crazy" combinations can be found at the end of this text.

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Notes:

Exercises 8.2

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.
2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.
3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.
4. T/F: Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

In Exercises 5–32, evaluate the indefinite integral.

5. $\int \sin^3 x \cos x \, dx$
6. $\int \cos^2 x \, dx$
7. $\int \cos^4 x \, dx$
8. $\int \sin^3 x \cos^2 x \, dx$
9. $\int \sin^3 x \cos^3 x \, dx$
10. $\int \sin^6 x \cos^5 x \, dx$
11. $\int \cos^2 x \tan^3 x \, dx$
12. $\int \sin^2 x \cos^2 x \, dx$
13. $\int \sin^3 x \sqrt{\cos x} \, dx$
14. $\int \sin(x) \cos(2x) \, dx$
15. $\int \sin(3x) \sin(7x) \, dx$
16. $\int \sin(\pi x) \sin(2\pi x) \, dx$
17. $\int \cos(x) \cos(2x) \, dx$
18. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x) \, dx$
19. $\int \tan^2 x \, dx$
20. $\int \tan^2 x \sec^4 x \, dx$
21. $\int \tan^3 x \sec^4 x \, dx$
22. $\int \tan^3 x \sec^2 x \, dx$
23. $\int \tan^3 x \sec^3 x \, dx$
24. $\int \tan^5 x \sec^5 x \, dx$
25. $\int \tan^4 x \, dx$
26. $\int \sec^5 x \, dx$
27. $\int \tan^2 x \sec x \, dx$
28. $\int \tan^2 x \sec^3 x \, dx$
29. $\int \csc x \, dx$
30. $\int \cot^3 x \csc^3 x \, dx$
31. $\int \cot^3 x \, dx$
32. $\int \cot^6 x \csc^4 x \, dx$

In Exercises 33–40, evaluate the definite integral.

33. $\int_0^{\pi} \sin x \cos^4 x \, dx$
34. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$
35. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$
36. $\int_0^{\pi/2} \sin(5x) \cos(3x) \, dx$
37. $\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x) \, dx$
38. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$
39. $\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$
40. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot^2 x \, dx$
41. Find the area between the curves $y = \sin^2 x$ and $y = \cos^2 x$ on the interval $[\pi/4, 3\pi/4]$.

8.3 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{9\pi}{2} \tag{8.3.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 5.5, though it can feel "backward." In Section 5.5, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

Watch the video:
Trigonometric Substitution — Example 3 / Part 1
at
https://youtu.be/yW6OduOYHLO

We start by demonstrating this method in evaluating the integral in Equation (8.3.1). After the example, we will generalize the method and give more examples.

Example 8.3.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\, dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\, d\theta$. We are almost ready to substitute. We also change our bounds of integration. The bound $x = -3$ corresponds to

Notes:

$\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}\int_{-3}^{3}\sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2}\sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}3\sqrt{9\cos^2\theta}\cos\theta\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}3\,|3\cos\theta|\cos\theta\,d\theta.\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a half-angle formula:

$$\begin{aligned}&= \int_{-\pi/2}^{\pi/2}9\cos^2\theta\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}\frac{9}{2}\big(1+\cos(2\theta)\big)\,d\theta\\ &= \frac{9}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.\end{aligned}$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples.

Key Idea 8.3.1 Trigonometric Substitution

(a) For integrands containing $\sqrt{a^2-x^2}$:
Let $x = a\sin\theta$, for $-\pi/2 \le \theta \le \pi/2$ and $a > 0$.
On this interval, $\cos\theta \ge 0$, so $\sqrt{a^2-x^2} = a\cos\theta$

(b) For integrands containing $\sqrt{x^2+a^2}$:
Let $x = a\tan\theta$, for $-\pi/2 < \theta < \pi/2$ and $a > 0$.
On this interval, $\sec\theta > 0$, so $\sqrt{x^2+a^2} = a\sec\theta$

(c) For integrands containing $\sqrt{x^2-a^2}$:
Let $x = a\sec\theta$, restricting our work to where $x \ge a > 0$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$.
On this interval, $\tan\theta \ge 0$, so $\sqrt{x^2-a^2} = a\tan\theta$

Notes:

Example 8.3.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,\mathrm{d}x$.

SOLUTION Using Key Idea 8.3.1(b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $\mathrm{d}x=\sqrt{5}\sec^2\theta\,\mathrm{d}\theta$. We will use the fact that $\sqrt{5+x^2}=\sqrt{5+5\tan^2\theta}=\sqrt{5\sec^2\theta}=\sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,\mathrm{d}\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,\mathrm{d}\theta\\ &= \int \sec\theta\,\mathrm{d}\theta\\ &= \ln\left|\sec\theta+\tan\theta\right|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The lengths of the sides of the reference triangle in Figure 8.3.1 are determined by the Pythagorean Theorem. With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta=\frac{x}{\sqrt{5}} \quad\text{and}\quad \sec\theta=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

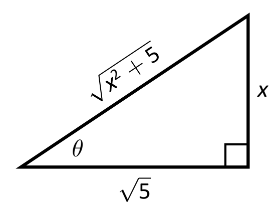


Figure 8.3.1: A reference triangle for Example 8.3.2

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,\mathrm{d}x &= \ln\left|\sec\theta+\tan\theta\right|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\big(1/\sqrt{5}\big)$ term is absorbed into the constant C. (In Section 7.4 we learned another way of approaching this problem.)

Notes:

Example 8.3.3 Using Trigonometric Substitution

Evaluate $\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it has the form $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a = 1/2$, and following Key Idea 8.3.1(c), we set $x = \frac{1}{2}\sec\theta$, and hence $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int \sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)d\theta\\ &= \int \sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int \sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int \frac{1}{2}\tan^2\theta\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^3\theta-\sec\theta\Big)\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 8.2.6, finding its antiderivatives to be

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Notes:

Thus

$$\int \sqrt{4x^2-1}\,dx$$
$$= \frac{1}{2}\int \left(\sec^3\theta - \sec\theta\right)d\theta$$
$$= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big) - \ln|\sec\theta+\tan\theta|\right) + C$$
$$= \frac{1}{4}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C.$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec\theta$, we use the Pythagorean Theorem to determine the lengths of the sides of the reference triangle in Figure 8.3.2.

$$\tan\theta = \frac{\sqrt{x^2-\frac{1}{4}}}{\frac{1}{2}} = 2\sqrt{x^2-\frac{1}{4}} \qquad \text{and} \qquad \sec\theta = 2x.$$

Therefore,

$$\int \sqrt{4x^2-1}\,dx = \frac{1}{4}\Big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\Big) + C$$
$$= \frac{1}{4}\left(2x\cdot 2\sqrt{x^2-\frac{1}{4}} - \ln\left|2x + 2\sqrt{x^2-\frac{1}{4}}\right|\right) + C$$
$$= \frac{1}{4}\left(4x\sqrt{x^2-\frac{1}{4}} - \ln\left|2x + 2\sqrt{x^2-\frac{1}{4}}\right|\right) + C$$
$$= \frac{1}{4}\left(2x\sqrt{4x^2-1} - \ln\left|2x + \sqrt{4x^2-1}\right|\right) + C.$$

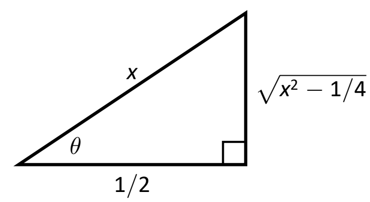


Figure 8.3.2: A reference triangle for Example 8.3.3

Example 8.3.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

SOLUTION We use Key Idea 8.3.1(a) with $a = 2$, $x = 2\sin\theta$, $dx =$

Notes:

$2\cos\theta d\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta \\ &= \int \cot^2\theta\,d\theta \\ &= \int(\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the Pythagorean Theorem we determine the lengths of the sides of the reference triangle in Figure 8.3.3. We have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

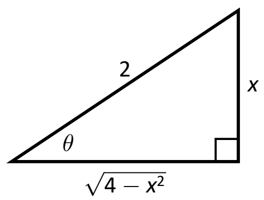


Figure 8.3.3: A reference triangle for Example 8.3.4

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 8.3.5 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1}x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 8.3.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta \\ &= \int 1\,d\theta \\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1}x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$.

The next example is similar to the previous one in that it does not involve a square-root. It shows how several techniques and identities can be combined to obtain a solution.

Notes:

Example 8.3.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$:

$$\begin{aligned} &= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta \\ &= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta \\ &= \int \cos^2\theta\,d\theta. \end{aligned}$$

Applying a half-angle formula, we have

$$\begin{aligned} &= \int \left(\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \qquad (8.3.2) \end{aligned}$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1}u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using a reference triangle, we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (8.3.2):

$$\begin{aligned} \frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)+C &= \frac{1}{2}\tan^{-1}u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C. \end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Note: Remember the sine and cosine double angle identities:

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$

They are often needed for writing your final answer in terms of x.

Notes:

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 8.3.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx$.

Solution Using Key Idea 8.3.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\begin{aligned}\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx &= \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta}5\sec^2\theta\,d\theta\\ &= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta.\end{aligned}$$

We encountered this indefinite integral in Example 8.3.3 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$\begin{aligned}25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta &= \frac{25}{2}\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}\\ &= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big).\end{aligned}$$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

Notes:

Exercises 8.3

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".
2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.
3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.
   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?
   (b) Use the new identity to simplify $9\tan^2\theta + 9$.
4. Why does Key Idea 8.3.1(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $|a\cos\theta|$?

Problems

In Exercises 5–26, apply Trigonometric Substitution to evaluate the indefinite integrals.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$
6. $\displaystyle\int \sqrt{x^2-1}\,dx$
7. $\displaystyle\int \sqrt{4x^2+1}\,dx$
8. $\displaystyle\int \sqrt{1-9x^2}\,dx$
9. $\displaystyle\int \sqrt{16x^2-1}\,dx$
10. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$
11. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$
12. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$
13. $\displaystyle\int \sqrt{x^2+4}\,dx$
14. $\displaystyle\int \sqrt{1-x^2}\,dx$
15. $\displaystyle\int \sqrt{9-x^2}\,dx$
16. $\displaystyle\int \sqrt{x^2-16}\,dx$
17. $\displaystyle\int \frac{7}{x^2+7}\,dx$
18. $\displaystyle\int \frac{3}{\sqrt{9-x^2}}\,dx$
19. $\displaystyle\int \frac{14}{\sqrt{5-x^2}}\,dx$
20. $\displaystyle\int \frac{2}{x\sqrt{x^2-9}}\,dx$
21. $\displaystyle\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$
22. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$
23. $\displaystyle\int \frac{1}{x^2-2x+8}\,dx$
24. $\displaystyle\int \frac{2}{\sqrt{-x^2+6x+7}}\,dx$
25. $\displaystyle\int \frac{3}{\sqrt{-x^2+8x+9}}\,dx$
26. $\displaystyle\int \frac{5}{x^2+6x+34}\,dx$

In Exercises 27–34, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

27. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$
28. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$
29. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$
30. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$
31. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$
32. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$
33. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$
34. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

In Exercises 35–40, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration.

35. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$
36. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\,dx$
37. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\,dx$
38. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\,dx$
39. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\,dx$
40. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$
41. Find the volume of the solid of revolution obtained by rotating the region bounded by $y=0$, $y=\frac{x}{\sqrt{1+x^2}}$, $x=0$, and $x=1$:
    (a) About the x-axis, using the disk method.
    (b) About the y-axis, using the shell method.

8.4 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where p and q do not have any common factors. We first consider the degree of p and q.

- If the $\deg(p) \geq \deg(q)$ then we use polynomial long division to divide q into p to determine a remainder $r(x)$ where $\deg(r) < \deg(q)$. We then write $f(x) = s(x) + \dfrac{r(x)}{q(x)}$ and apply partial fraction decomposition to $\dfrac{r(x)}{q(x)}$.
- If the $\deg(p) < \deg(q)$ we can apply partial fraction decomposition to $\dfrac{p(x)}{q(x)}$ without additional work.

Partial fraction decomposition is based on an algebraic theorem that guarantees that any polynomial, and hence q, can use real numbers to factor into the product of linear and irreducible quadratic factors. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

An *irreducible quadratic* is one that cannot factor into linear terms with real coefficients.

Notes:

Key Idea 8.4.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where $\deg(p) < \deg(q)$.

1. **Factor $\mathbf{q(x)}$** : Write $q(x)$ as the product of its linear and irreducible quadratic factors of the form $(ax+b)^m$ and $(ax^2+bx+c)^n$ where m and n are the highest powers of each factor that divide q.

   - **Linear Terms:** For each linear factor of $q(x)$ the decomposition of $\dfrac{p(x)}{q(x)}$ will contain the following terms:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_m}{(ax+b)^m}$$

   - **Irreducible Quadratic Terms:** For each irreducible quadratic factor of $q(x)$ the decomposition of $\dfrac{p(x)}{q(x)}$ will contain the following terms:

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

2. **Finding the Coefficients $\mathbf{A_i}$, $\mathbf{B_i}$, and $\mathbf{C_i}$:**

   - Set $\dfrac{p(x)}{q(x)}$ equal to the sum of its linear and irreducible quadratic terms.

$$\frac{p(x)}{q(x)} = \frac{A_1}{(ax+b)} + \cdots \frac{A_m}{(ax+b)^m} + \frac{B_1x+C_1}{(ax^2+bx+c)} + \cdots \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

   - Multiply this equation by the factored form of $q(x)$ and simplify to clear the denominators.
   - Solve for the coefficients A_i, B_i, and C_i by
     (a) multiplying out the remaining terms and collecting like powers of x, equating the resulting coefficients and solving the resulting system of linear equations, **or**
     (b) substituting in values for x that eliminate terms so the simplified equation can be solved for a coefficient.

Notes:

Watch the video:
Integration Using method of Partial Fractions at
https://youtu.be/6qVgHWxdlZ0

The following examples will demonstrate how to put this Key Idea into practice. In Example 8.4.1, we focus on the setting up the decomposition of a rational function.

Example 8.4.1 Decomposing into partial fractions

Decompose $f(x) = \dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 are irreducible quadratics. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear factor that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2},\quad \frac{C}{(x-2)^2}\quad \text{and}\quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7}\quad \text{and}\quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\begin{aligned}\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} &= \\ &\frac{A}{x+5}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}+ \\ &\frac{Ex+F}{x^2+x+2}+\frac{Gx+H}{x^2+x+7}+\frac{Ix+J}{(x^2+x+7)^2}\end{aligned}$$

Notes:

Solving for the coefficients A, B, ..., J would be a bit tedious but not "hard." In the next example we demonstrate solving for the coefficients using both methods given in Key Idea 8.4.1.

Example 8.4.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator can be written as the product of two linear factors: $x^2-1=(x-1)(x+1)$. Thus

$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}. \tag{8.4.1}$$

Using the method described in Key Idea 8.4.1 2(a) to solve for A and B, first multiply through by $x^2-1=(x-1)(x+1)$:

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x-1}+\frac{B(x-1)(x+1)}{x+1} \\ &= A(x+1)+B(x-1) \qquad\qquad (8.4.2) \\ &= Ax+A+Bx-B \\ &= (A+B)x+(A-B) \qquad \text{collect like terms.} \end{aligned}$$

The next step is key. For clarity's sake, rewrite the equality we have as

$$0x+1=(A+B)x+(A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal for all values of x, we must have that $0=A+B$. Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A-B)$. Therefore we have $1=A-B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{aligned} A+B&=0 \\ A-B&=1 \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A&=1/2 \\ B&=-1/2. \end{aligned}$$

Thus

$$\frac{1}{x^2-1}=\frac{1/2}{x-1}-\frac{1/2}{x+1}.$$

Before solving for A and B using the method described in Key Idea 8.4.1 2(b), we note that Equations (8.4.1) and (8.4.2) are not equivalent. Only the second equation holds for all values of x, including $x=-1$ and $x=1$, by continuity

Notes:

of polynomials. Thus, we can choose values for x that eliminate terms in the polynomial to solve for A and B.

$$1 = A(x+1) + B(x-1).$$

If we choose $x = -1$,

$$\begin{aligned} 1 &= A(0) + B(-2) \\ B &= -\frac{1}{2}. \end{aligned}$$

Next choose $x = 1$:

$$\begin{aligned} 1 &= A(2) + B(0) \\ A &= \frac{1}{2}. \end{aligned}$$

Resulting in the same decomposition as above.

In Example 8.4.3, we solve for the decomposition coefficients using the system of linear equations (method 2a). The margin note explains how to solve using substitution (method 2b).

Example 8.4.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

SOLUTION We decompose the integrand as follows, as described by Key Idea 8.4.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}. \tag{8.4.3}$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (8.4.4) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

Note: Equations (8.4.3) and (8.4.4) are not equivalent for $x = 1$ and $x = -2$. However, due to the continuity of polynomials we can let $x = 1$ to simplify the right hand side to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$, so that $A = 1/9$.

Likewise,when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.

Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

Notes:

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$. The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 8.4.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Solution Key Idea 8.4.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x + 2 + \frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 8.4.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$19x + 30 = A(x+3) + B(x-5).$$

As in the previous examples we choose values of x to eliminate terms in the polynomial. If we choose $x = -3$,

$$\begin{aligned} 19(-3) + 30 &= A(0) + B(-8) \\ B &= \frac{27}{8}. \end{aligned}$$

Next choose $x = 5$:

$$\begin{aligned} 19(5) + 30 &= A(8) + B(0) \\ A &= \frac{125}{8}. \end{aligned}$$

Notes:

We can now integrate:

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right) dx\\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C.\end{aligned}$$

Before the next example we remind the reader of a rational integrand evaluated by trigonometric substitution:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C.$$

Example 8.4.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 8.4.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$7x^2+31x+54 = A(x^2+6x+11)+(Bx+C)(x+1).$$

Again, we choose values of x to eliminate terms in the polynomial. If we choose $x=-1$,

$$\begin{aligned}30 &= 6A+(-B+C)(0)\\ A &= 5.\end{aligned}$$

Although none of the other terms can be zeroed out, we continue by letting $A=5$ and substituting helpful values of x. Choosing $x=0$, we notice

$$\begin{aligned}54 &= 55+C\\ C &= -1.\end{aligned}$$

Finally, choose $x=1$ (any value other than -1 and 0 can be used, 1 is easy to work with)

$$\begin{aligned}92 &= 90+(B-1)(2)\\ B &= 2.\end{aligned}$$

Notes:

Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right) dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u = x^2+6x+11$, so $du = (2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11}\\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, yielding $\ln\left|x^2+6x+11\right|$. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Key Idea 8.3.1 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}&\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx\\ &\quad= \int \left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right) dx\\ &\quad= \int \frac{5}{x+1}\,dx + \int \frac{2x+6}{x^2+6x+11}\,dx - \int \frac{7}{x^2+6x+11}\,dx\\ &\quad= 5\ln|x+1| + \ln\left|x^2+6x+11\right| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Notes:

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals. The next section will require the reader to determine an appropriate method for evaluating a variety of integrals.

Notes:

Exercises 8.4

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.
2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.
3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 8.4.1.
4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 8.4.1.
5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 8.4.1.
6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 8.4.1.

Problems

In Exercises 7–34, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$
8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$
9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$
10. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$
11. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$
12. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$
13. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$
14. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$
15. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$
16. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$
17. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$
18. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$
19. $\displaystyle\int \frac{dx}{x^4-x^2}$
20. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$
21. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$
22. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$
23. $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$
24. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$
25. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$
26. $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$
27. $\displaystyle\int \frac{x}{x^4+4x^2+3}\,dx$
28. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}\,dx$
29. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$
30. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$
31. $\displaystyle\int \frac{1}{x^4-16}\,dx$
32. $\displaystyle\int \frac{1}{x^2+x}\,dx$
33. $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$
34. $\displaystyle\int \frac{2x^2}{(x^2+1)^2}\,dx$

In Exercises 35–38, evaluate the definite integral.

35. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$
36. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$
37. $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$
38. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$
39. Recall that
$$\frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{x^2+1}}$$
Now use a trigonometric substitution to evaluate the indefinite integral
$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$
and show that
$$\sinh^{-1}x=\ln(x+\sqrt{x^2+1}).$$

8.5 Integration Strategies

We've now seen a fair number of different integration techniques and so we should probably pause at this point to talk a little bit about a strategy to use for determining the correct technique to use when faced with an integral.

There are a couple of points that need to be made about this strategy. First, it isn't a hard and fast set of rules for determining the method that should be used. It is really nothing more than a general set of guidelines that will help us to identify techniques that may work. Some integrals can be done in more than one way and so depending on the path you take through the strategy you may end up with a different technique than someone else who also went through this strategy.

Second, while the strategy is presented as a way to identify the technique that could be used on an integral keep in mind that, for many integrals, it can also automatically exclude certain techniques as well. When going through the strategy keep two lists in mind. The first list is integration techniques that simply won't work and the second list is techniques that look like they might work. After going through the strategy, if the second list has only one entry then that is the technique to use. If on the other hand, there is more than one possible technique to use we will have to decide on which is liable to be the best for us to use. Unfortunately there is no way to teach which technique is the best as that usually depends upon the person and which technique they find to be the easiest.

Third, don't forget that many integrals can be evaluated in multiple ways and so more than one technique may be used on it. This has already been mentioned in each of the previous points, but is important enough to warrant a separate mention. Sometimes one technique will be significantly easier than the others and so don't just stop at the first technique that appears to work. Always identify all possible techniques and then go back and determine which you feel will be the easiest for you to use.

Next, it's entirely possible that you will need to use more than one method to completely evaluate an integral. For instance a substitution may lead to using integration by parts or partial fractions integral.

Notes:

Key Idea 8.5.1 Guidelines for Choosing an Integration Strategy

1. Simplify the integrand, if possible.
2. See if a "simple" substitution will work.
3. Identify the type of integral.
4. Relate the integral to an integral we already know how to do.
5. Try multiple techniques.
6. Try again.

Let's expand on the ideas of the previous Key Idea.

1. **Simplify the integrand, if possible.** This step is very important in the integration process. Many integrals can be taken from very difficult to very easy with a little simplification or manipulation. Don't forget basic trigonometric and algebraic identities as these can often be used to simplify the integral.

   We used this idea when we were looking at integrals involving trigonometric functions. For example consider the integral

   $$\int \cos^2 x \, \mathrm{d}x.$$

   The integral can't be done as it is, however by recalling the identity,

   $$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

   the integral becomes very easy to do.

   Note that this example also shows that simplification does not necessarily mean that we'll write the integrand in a "simpler" form. It only means that we'll write the integrand in a form that we can deal with and this is often longer and/or "messier" than the original integral.

2. **See if a "simple" substitution will work.** Look to see if a simple substitution can be used instead of the often more complicated methods from this chapter. For example consider both of the following integrals.

   $$\int \frac{x}{x^2 - 1} \, \mathrm{d}x \qquad \int x\sqrt{x^2 - 1} \, \mathrm{d}x$$

Notes:

The first integral can be done with the method of partial fractions and the second could be done with a trigonometric substitution.

However, both could also be evaluated using the substitution $u = x^2 - 1$ and the work involved in the substitution would be significantly less than the work involved in either partial fractions or trigonometric substitution.

So, always look for quick, simple substitutions before moving on to the more complicated techniques of this chapter.

3. **Identify the type of integral.** Note that any integral may fall into more than one of these types. Because of this fact it's usually best to go all the way through the list and identify all possible types since one may be easier than the other and it's entirely possible that the easier type is listed lower in the list.

   (a) Is the integrand a rational expression (i.e. is the integrand a polynomial divided by a polynomial)? If so then partial fractions (Section 8.4) may work on the integral.

   (b) Is the integrand a polynomial times a trigonometric function, exponential, or logarithm? If so, then integration by parts (Section 8.1) may work.

   (c) Is the integrand a product of sines and cosines, secants and tangents, or cosecants and cotangents? If so, then the topics from Section 8.2 may work. Likewise, don't forget that some quotients involving these functions can also be done using these techniques.

   (d) Does the integrand involve $\sqrt{b^2x^2 + a^2}$, $\sqrt{b^2x^2 - a^2}$, or $\sqrt{a^2 - b^2x^2}$? If so, then a trigonometric substitution (Section 8.3) might work nicely.

   (e) Does the integrand have roots other than those listed above in it? If so then the substitution $u = \sqrt[n]{g(x)}$ might work.

   (f) Does the integrand have a quadratic in it? If so then completing the square on the quadratic might put it into a form that we can deal with.

4. **Relate the integral to an integral we already know how to do.** In other words, can we use a substitution or manipulation to write the integrand into a form that does fit into the forms we've looked at previously in this chapter. A typical example is the following integral.

$$\int \cos x\sqrt{1 + \sin^2 x}\,dx$$

Notes:

This integral doesn't obviously fit into any of the forms we looked at in this chapter. However, with the substitution $u = \sin x$ we can reduce the integral to the form

$$\int \sqrt{1+u^2}\,dx$$

which is a trigonometric substitution problem.

5. **Try multiple techniques.** In this step we need to ask ourselves if it is possible that we'll need to use multiple techniques. The example in the previous part is a good example. Using a substitution didn't allow us to actually do the integral. All it did was put the integral into a form that we could use a different technique on.

   Don't ever get locked into the idea that an integral will only require one step to completely evaluate it. Many will require more than one step.

6. **Try again.** If everything that you've tried to this point doesn't work then go back through the process again. This time try a technique that you didn't use the first time around.

As noted above, this strategy is not a hard and fast set of rules. It is only intended to guide you through the process of best determining how to do any given integral. Note as well that the only place Calculus II actually arises is the third step. Steps 1, 2, and 4 involve nothing more than manipulation of the integrand either through direct manipulation of the integrand or by using a substitution. The last two steps are simply ideas to think about in going through this strategy.

Many students go through this process and concentrate almost exclusively on Step 3 (after all this is Calculus II, so it's easy to see why they might do that...) to the exclusion of the other steps. One very large consequence of that exclusion is that often a simple manipulation or substitution is overlooked that could make the integral very easy to do.

Before moving on to the next section we will work a couple of examples illustrating a couple of not so obvious simplifications/manipulations and a not so obvious substitution.

Example 8.5.1 Strategies of Integration

Evaluate the integral

$$\int \frac{\tan x}{\sec^4 x}\,dx$$

Solution This integral almost falls into the form given in 3c. It is a quotient of tangent and secant and we know that sometimes we can use the same methods for products of tangents and secants on quotients.

Notes:

The process from Section 8.2 tells us that if we have even powers of secant to save two of them and convert the rest to tangents. That won't work here. We can save two secants, but they would be in the denominator and they won't do us any good here. Remember that the point of saving them is so they could be there for the substitution $u = \tan x$. That requires them to be in the numerator. So, that won't work. We need to find another solution method.

There are in fact two solution methods to this integral depending on how you want to go about it.

Solution 1 In this solution method we could just convert everything to sines and cosines and see if that gives us an integral we can deal with.

$$\begin{aligned}\int \frac{\tan x}{\sec^4 x}\,dx &= \int \frac{\sin x}{\cos x}\cos^4 x\,dx \\ &= \int \sin x \cos^3 x\,dx \qquad \text{substitute } u = \cos x \\ &= -\int u^3\,du \\ &= -\frac{1}{4}\cos^4 x + C\end{aligned}$$

Note that just converting to sines and cosines won't always work and if it does it won't always work this nicely. Often there will be a lot more work that would need to be done to complete the integral.

Solution 2 This solution method goes back to dealing with secants and tangents. Let's notice that if we had a secant in the numerator we could just use $u = \sec x$ as a substitution and it would be a fairly quick and simple substitution to use. We don't have a secant in the numerator. However we could very easily get a secant in the numerator by multiplying the numerator and denominator by secant (i.e. we multiply the integrand by "1").

$$\begin{aligned}\int \frac{\tan x}{\sec^4 x}\,dx &= \int \frac{\tan x \sec x}{\sec^5 x}\,dx \qquad \text{substitute } u = \sec x \\ &= \int \frac{1}{u^5}\,du \\ &= -\frac{1}{4}\frac{1}{\sec^4 x} + C \\ &= -\frac{1}{4}\cos^4 x + C\end{aligned}$$

In the previous example we saw two "simplifications" that allowed us to evaluate the integral. The first was using identities to rewrite the integral into terms

Notes:

we could deal with and the second involved multiplying the numerator and denominator by something to again put the integral into terms we could deal with.

Using identities to rewrite an integral is an important "simplification" and we should not forget about it. Integrals can often be greatly simplified or at least put into a form that can be dealt with by using an identity.

The second "simplification" is not used as often, but does show up on occasion so again, it's best to remember it. In fact, let's take another look at an example in which multiplying the integrand by "1" will allow us to evaluate an integral.

Example 8.5.2 Strategy for Integration

Evaluate the integral

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x$$

Solution This is an integral which if we just concentrate on the third step we won't get anywhere. This integral doesn't appear to be any of the kinds of integrals that we worked on in this chapter. We can evaluate the integral however, if we do the following,

$$\begin{aligned}\int \frac{1}{1+\sin x}\,\mathrm{d}x &= \int \frac{1}{1+\sin x}\frac{1-\sin x}{1-\sin x}\,\mathrm{d}x\\ &= \int \frac{1-\sin x}{1-\sin^2 x}\,\mathrm{d}x\end{aligned}$$

This does not appear to have done anything for us. However, if we now remember the first "simplification" we looked at above we will notice that we can use an identity to rewrite the denominator. Once we do that we can further manipulate the integrand into something we can evaluate.

$$\begin{aligned}\int \frac{1}{1+\sin x}\,\mathrm{d}x &= \int \frac{1-\sin x}{\cos^2 x}\,\mathrm{d}x\\ &= \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x}\frac{1}{\cos x}\,\mathrm{d}x\\ &= \int \sec^2 x - \tan x \sec x\,\mathrm{d}x\\ &= \tan x - \sec x + C\end{aligned}$$

So, we've just seen once again that multiplying by a helpful form of "1" can put the integral into a form we can integrate. Notice as well that this example also showed that "simplifications" do not necessarily put an integral into a simpler form. They only put the integrand into a form that is easier to integrate.

Notes:

Let's now take a quick look at an example of a substitution that is not so obvious.

Example 8.5.3 Strategy for Integration

Evaluate the integral

$$\int \cos\sqrt{x}\,\mathrm{d}x$$

SOLUTION We introduced this integral by saying that the substitution was not so obvious. However, this is really an integral that falls into the form given by 3e in Key Idea 8.5.1. Many people miss that form and so don't think about it. So, let's try the following substitution.

$$u = \sqrt{x} \qquad x = u^2 \qquad \mathrm{d}x = 2u\,\mathrm{d}u$$

With this substitution the integral becomes,

$$\int \cos\sqrt{x}\,\mathrm{d}x = 2\int u\cos u\,\mathrm{d}u$$

This is now an integration by parts. Remember that often we will need to use more than one technique to completely do the integral. This is a fairly simple integration by parts problem so we'll leave the remainder of the details for you to check.

$$\int \cos\sqrt{x}\,\mathrm{d}x = 2(\cos\sqrt{x} + \sqrt{x}\sin\sqrt{x}) + C.$$

It will be possible to integrate every integral assigned in this class, but it is important to note that there are integrals that just can't be evaluated. We should also note that after we look at series in Chapter 9 we will be able to write down a series representation of many of these types of integrals.

Notes:

Exercises 8.5

Problems

In Exercises 1–52, compute the indefinite integral.

1. $\int \sin^{-1} x \, dx$

2. $\int \cos^3 2x \, \sin^2 2x \, dx$

3. $\int \dfrac{4x^2 - 12x - 10}{(x-2)(x^2 - 4x + 3)} \, dx$

4. $\int \tan x \, \sec^5 x \, dx$

5. $\int \dfrac{1}{(x^2 + 25)^{3/2}} \, dx$

6. $\int \dfrac{\sqrt{4 - x^2}}{x} \, dx$

7. $\int \dfrac{x^3 + 1}{x(x-1)^3} \, dx$

8. $\int \dfrac{x}{\sqrt{4 + 4x - x^2}} \, dx$

9. $\int x^3 e^{x^2} \, dx$

10. $\int \dfrac{\sqrt[3]{x + 8}}{x} \, dx$

11. $\int e^{2x} \sin 3x \, dx$

12. $\int \cos^3 x \, \sin^3 x \, dx$

13. $\int \dfrac{x}{\sqrt{4 - x^2}} \, dx$

14. $\int \dfrac{x^5 - x^3 + 1}{x^3 + 2x^2} \, dx$

15. $\int \dfrac{1}{x^{3/2} + x^{1/2}} \, dx$

16. $\int e^x \sec e^x \, dx$

17. $\int x^2 \sin 3x \, dx$

18. $\int \sin^3 x \sqrt{\cos x} \, dx$

19. $\int e^x \sqrt{e^x + 1} \, dx$

20. $\int \dfrac{x^2}{\sqrt{4x^2 + 9}} \, dx$

21. $\int \sec^2 x \tan^2 x \, dx$

22. $\int x \csc x \cot x \, dx$

23. $\int x^2 (8 - x^3)^{1/3} \, dx$

24. $\int \sin \sqrt{x} \, dx$

25. $\int x\sqrt{3 - 2x} \, dx$

26. $\int \dfrac{e^{3x}}{1 + e^x} \, dx$

27. $\int \dfrac{x^2 - 4x + 3}{\sqrt{x}} \, dx$

28. $\int \dfrac{x^3}{\sqrt{16 - x^2}} \, dx$

29. $\int \dfrac{1 - 2x}{x^2 + 12x + 35} \, dx$

30. $\int \tan^{-1} 5x \, dx$

31. $\int \dfrac{e^{\tan x}}{\cos^2 x} \, dx$

32. $\int \dfrac{1}{\sqrt{7 + 5x^2}} \, dx$

33. $\int \cot^6 x \, dx$

34. $\int x^3 \sqrt{x^2 - 25} \, dx$

35. $\int (x^2 - \operatorname{sech}^2 4x) \, dx$

36. $\int x^2 e^{-4x} \, dx$

37. $\int \dfrac{3}{\sqrt{11 - 10x - x^2}} \, dx$

38. $\int \dfrac{x^3 - 20x^2 - 63x - 198}{x^4 - 1} \, dx$

39. $\int \tan^3 x \sec x \, dx$

40. $\int (x^3 + 1) \cos x \, dx$

41. $\int \dfrac{\sqrt{9 - 4x^2}}{x^2} \, dx$

42. $\int (5 - \cot 3x)^2 \, dx$

43. $\int \dfrac{1}{x(\sqrt{x} - \sqrt[4]{x})} \, dx$

44. $\int \dfrac{\sin x}{\sqrt{1 + \cos x}} \, dx$

45. $\int \dfrac{x^2}{(25 + x^2)^2} \, dx$

46. $\int \dfrac{2x^3 + 4x^2 + 10x + 13}{x^4 + 9x^2 + 20} \, dx$

47. $\int \dfrac{(x^2 - 2)^2}{x} \, dx$

48. $\int x^{3/2} \ln x \, dx$

49. $\int \dfrac{x^2}{\sqrt[3]{2x + 3}} \, dx$

50. $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx$

51. $\displaystyle\int \tan 7x \cos 7x\,dx$

52. $\displaystyle\int x \sin^{-1} x\,dx$

53. Show that if $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \quad \text{and} \quad \frac{d\theta}{dt} = \frac{2}{1+t^2}.$$

Explain how this substitution transforms a rational trigonometric function into a rational function. (This is known as the *tangent half-angle substitution* and sometimes known as the Weierstrass substitution.)

54. Use the substitution of the previous exercise to evaluate $\displaystyle\int \sec\theta\,d\theta$.

55. Use the substitution of the previous exercise to evaluate $\displaystyle\int \frac{1}{\sin\theta + \tan\theta}\,d\theta$.

8.6 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 8.6.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^b = \tan^{-1}b - \tan^{-1}0 = \tan^{-1}b.$$

As $b \to \infty$, $\tan^{-1}b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a,b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a,b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

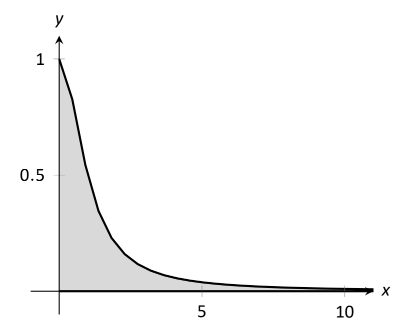


Figure 8.6.1: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notes:

Improper Integrals with Infinite Bounds

Definition 8.6.1 Improper Integrals with Infinite Bounds

1. Let f be a continuous function on $[a, \infty)$. For $t \geq a$ let
$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.$$
2. Let f be a continuous function on $(-\infty, b]$. For $t \leq b$ let
$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x.$$
3. Let f be a continuous function on $(-\infty, \infty)$. For any real number c (which one doesn't matter), let
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^c f(x)\,\mathrm{d}x \;+\; \lim_{b\to\infty}\int_c^b f(x)\,\mathrm{d}x.$$

An improper integral is said to **converge** if its corresponding limit exists and is finite; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Watch the video:
Improper Integral — Infinity in Upper and Lower Limits at
`https://youtu.be/f6cGotvktxs`

Example 8.6.1 Evaluating improper integrals
Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$
3. $\displaystyle\int_{-\infty}^0 e^x\,\mathrm{d}x$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x$

Notes:

Solution

1. $$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty} \frac{-1}{x}\Bigg|_1^t\\ &= \lim_{t\to\infty} \frac{-1}{t}+1\\ &= 1.\end{aligned}$$
   A graph of the area defined by this integral is given in Figure 8.6.2.

2. $$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx\\ &= \lim_{t\to\infty} \ln|x|\Big|_1^t\\ &= \lim_{t\to\infty} \ln(t)\\ &= \infty.\end{aligned}$$
   The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 8.6.2 and 8.6.3; notice how the values of $f(x) = 1/x$ are noticeably larger than those of $f(x) = 1/x^2$. This difference is enough to cause the improper integral to diverge.

3. $$\begin{aligned}\int_{-\infty}^0 e^x\,dx &= \lim_{t\to-\infty}\int_t^0 e^x\,dx\\ &= \lim_{t\to-\infty} e^x\Big|_t^0\\ &= \lim_{t\to-\infty} \left(e^0 - e^t\right)\\ &= 1.\end{aligned}$$
   A graph of the area defined by this integral is given in Figure 8.6.4.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 8.6.1. Any value of c is fine; we choose $c = 0$.

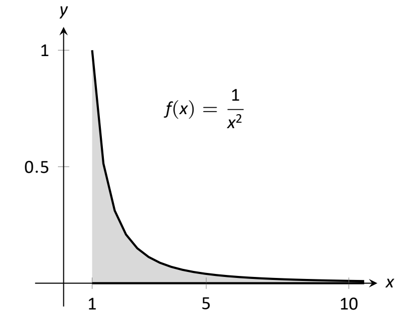


Figure 8.6.2: A graph of $f(x) = \frac{1}{x^2}$ in Example 8.6.1 part 1.

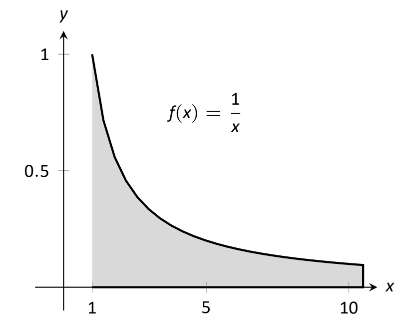


Figure 8.6.3: A graph of $f(x) = \frac{1}{x}$ in Example 8.6.1 part 2.

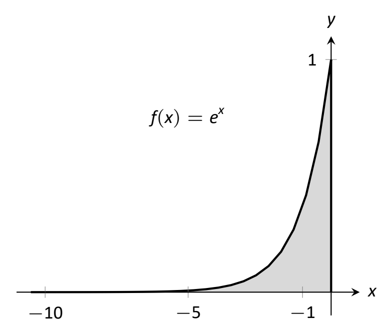


Figure 8.6.4: A graph of $f(x) = e^x$ in Example 8.6.1 part 3.

Notes:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx$$
$$= \lim_{t\to-\infty} \tan^{-1} x\Big|_t^0 + \lim_{t\to\infty} \tan^{-1} x\Big|_0^t$$
$$= \lim_{t\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} t\right) + \lim_{t\to\infty}\left(\tan^{-1} t - \tan^{-1} 0\right)$$
$$= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).$$
$$= \pi.$$

A graph of the area defined by this integral is given in Figure 8.6.5.

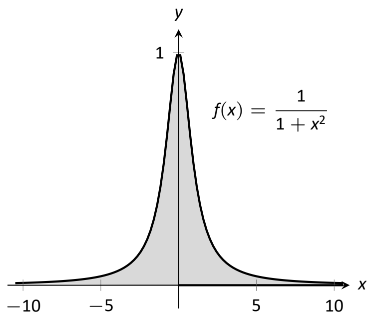


Figure 8.6.5: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 8.6.1 part 4.

Section 7.5 introduced L'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Example 8.6.2 Improper integration and L'Hôpital's Rule

Evaluate the improper integral $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx$$
$$= \lim_{t\to\infty}\left(-\frac{\ln x}{x}\Big|_1^t + \int_1^t \frac{1}{x^2}\,dx\right)$$
$$= \lim_{t\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^t$$
$$= \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} - \left(-\ln 1 - 1\right)\right).$$

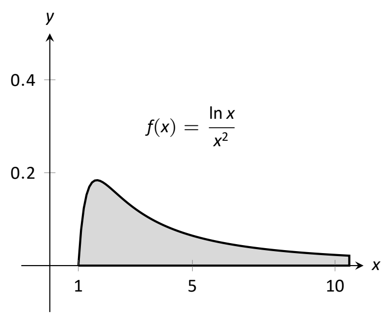


Figure 8.6.6: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 8.6.2.

The $1/t$ goes to 0, and $\ln 1 = 0$, leaving $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t}$ with L'Hôpital's Rule. We have:

$$\lim_{t\to\infty}\frac{\ln t}{t} \overset{\text{by LHR}}{=} \lim_{t\to\infty}\frac{1/t}{1} = 0.$$

Thus the improper integral evaluates as:

$$\int_1^{\infty}\frac{\ln x}{x^2}\,dx = 1.$$

Notes:

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 8.6.2 Improper Integration with Infinite Range
Let $f(x)$ be a continuous function on $[a, b]$ except at c, $a \leq c \leq b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx.$$

Note that c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Example 8.6.3 Improper integration of functions with infinite range
Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ 2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

Solution

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 8.6.7. Notice that f has a vertical asymptote at $x = 0$. In some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{t \to 0^+} 2\sqrt{x}\Big|_t^1\\ &= \lim_{t \to 0^+} 2\left(\sqrt{1} - \sqrt{t}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 8.6.8, so this integral is an improper integral. Let's eschew using

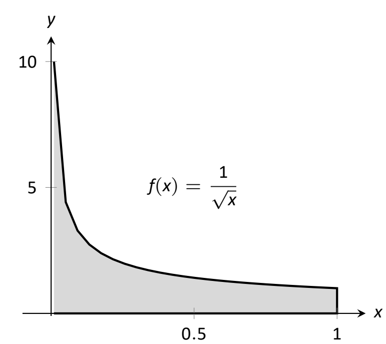


Figure 8.6.7: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 8.6.3.

Notes:

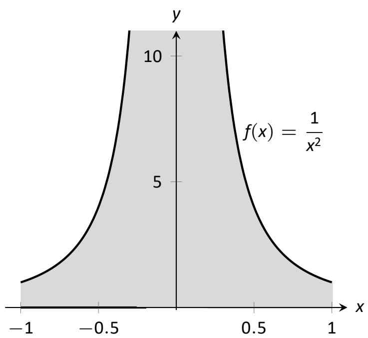


Figure 8.6.8: A graph of $f(x) = \frac{1}{x^2}$ in Example 8.6.3.

limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^{1} \\ &= -1-(1) \\ &= -2.\end{aligned}$$

Clearly the area in question is above the x-axis, yet the area is supposedly negative. In this example we noted the discontinuity of the integrand on $[-1,1]$ (its improper nature) but continued anyway to apply the Fundamental Theorem of Calculus. Violating the hypothesis of the FTC led us to an incorrect area of -2. If we now evaluate the integral using Definition 8.6.2 we will see that the area is unbounded.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^{t}\frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_{t}^{1}\frac{1}{x^2}\,dx \\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{1} \\ &= \lim_{t\to 0^-}\left(-\frac{1}{t}-1\right) + \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is knowing the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 8.6.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^{\infty}\frac{1}{x^p}\,dx$ converges.

Notes:

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx \\ &= \lim_{t\to\infty}\int_1^t x^{-p}\,dx \qquad \text{(assume } p\neq 1\text{)} \\ &= \lim_{t\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^t \\ &= \lim_{t\to\infty}\frac{1}{1-p}\big(t^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge — i.e., when is this limit *not* ∞? This limit converges precisely when the power of t is less than 0: when $1-p<0 \Rightarrow 1<p$.

Our analysis shows that if $p>1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p<1$ the improper integral diverges; we showed in Example 8.6.1 that when $p=1$ the integral also diverges.

Figure 8.6.9 graphs $y=1/x$ with a dashed line, along with graphs of $y=1/x^p$, $p<1$, and $y=1/x^q$, $q>1$. Somehow the dashed line forms a dividing line between convergence and divergence.

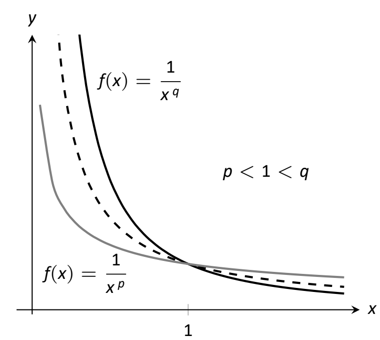


Figure 8.6.9: Plotting functions of the form $1/x^p$ in Example 8.6.4.

The result of Example 8.6.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 8.6.1 Convergence of Improper Integrals $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$ and diverges when $p\leq 1$.
2. The improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges when $p<1$ and diverges when $p\geq 1$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose

Notes:

convergence is known. We often use integrands of the form $1/x^p$ in comparisons as their convergence on certain intervals is known. This is described in the following theorem.

Note: We used the upper and lower bound of "1" in Key Idea 8.6.1 for convenience. It can be replaced by any a where $a > 0$.

Theorem 8.6.1 Direct Comparison Test for Improper Integrals
Let f and g be continuous on $[a, \infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$.

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example 8.6.5 Determining convergence of improper integrals
Determine the convergence of the following improper integrals.

$$1.\ \int_1^\infty e^{-x^2}\,dx \qquad\qquad 2.\ \int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 8.6.10, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 8.6.1 that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^\infty e^{-x^2}\,dx$ also converges.

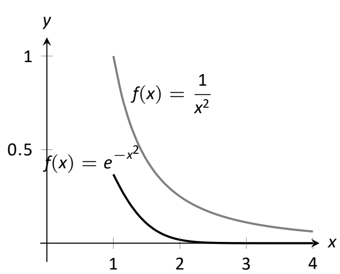


Figure 8.6.10: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 8.6.5.

2. Note that for large values of x, $\dfrac{1}{\sqrt{x^2-x}} \approx \dfrac{1}{\sqrt{x^2}} = \dfrac{1}{x}$. We know from Key Idea 8.6.1 and the subsequent note that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving

$$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$

Notes:

Using Theorem 8.6.1, we conclude that since $\int_3^\infty \frac{1}{x}\,dx$ diverges, then $\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 8.6.11 illustrates this.

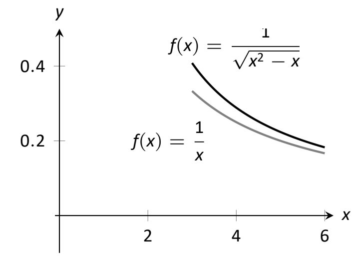


Figure 8.6.11: Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 8.6.5.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\frac{1}{x} > \frac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 8.6.1.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 8.6.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a,\infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 8.6.6 Determining convergence of improper integrals

Determine the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the square root of a quadratic function will begin to behave much like $y = x$. So we compare $\frac{1}{\sqrt{x^2+2x+5}}$ to $\frac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using L'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ L'Hôpital's Rule at least once to verify this.)

Notes:

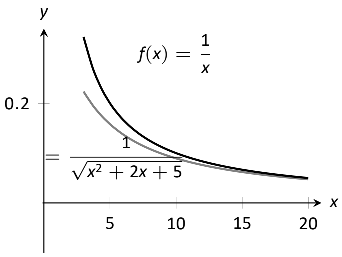


Figure 8.6.12: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 8.6.6.

The trouble is the square root function. We determine the limit by using a technique we learned in Key Idea 1.5.1:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}} = \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{x^2+2x+5}{x^2}}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{2}{x}+\frac{5}{x^2}}} = 1$$

Since we know that $\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 8.6.12 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Integration by Parts, which reverses the Product Rule of differentiation. We also learned specialized techniques for handling trigonometric and rational functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

Notes:

Exercises 8.6

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim_{b\to\infty} \int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to ________________.

3. If $\int_1^\infty f(x)\,dx = 10$, and $0 \le g(x) \le f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ ________________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Problems

In Exercises 7–36, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\,dx$

8. $\int_1^\infty \frac{1}{x^3}\,dx$

9. $\int_1^\infty x^{-4}\,dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

11. $\int_{-\infty}^0 2^x\,dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

14. $\int_{-\infty}^\infty \frac{x}{x^2+4}\,dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\,dx$

17. $\int_2^\infty \frac{1}{x-1}\,dx$

18. $\int_1^2 \frac{1}{x-1}\,dx$

19. $\int_0^3 \frac{1}{x}\,dx$

20. $\int_{-1}^1 \frac{1}{x}\,dx$

21. $\int_2^5 \frac{dx}{\sqrt{x-2}}$

22. $\int_1^9 \frac{dx}{\sqrt[3]{9-x}}$

23. $\int_1^3 \frac{1}{x-2}\,dx$

24. $\int_0^\pi \sec^2 x\,dx$

25. $\int_0^{\frac{\pi}{2}} \sec x\,dx$

26. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

27. $\int_0^\infty xe^{-x}\,dx$

28. $\int_0^\infty xe^{-x^2}\,dx$

29. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

30. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

31. $\int_0^1 x\ln x\,dx$

32. $\int_1^\infty \frac{\ln x}{x}\,dx$

33. $\int_0^1 \ln x\,dx$

34. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

35. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

36. $\int_0^\infty e^{-x}\sin x\,dx$

In Exercises 37–46, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

37. $\int_{10}^\infty \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

38. $\int_2^\infty \frac{4}{\sqrt{7x^3-x}}\,dx$

39. $\int_0^\infty \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

40. $\int_1^\infty e^{-x}\ln x\,dx$

41. $\int_5^\infty e^{-x^2+3x+1}\,dx$

42. $\int_0^\infty \frac{\sqrt{x}}{e^x}\,dx$

43. $\int_2^\infty \frac{1}{x^2+\sin x}\,dx$

44. $\int_0^\infty \frac{x}{x^2+\cos x}\,dx$

45. $\displaystyle\int_0^\infty \frac{1}{x+e^x}\,dx$

46. $\displaystyle\int_0^\infty \frac{1}{e^x-x}\,dx$

47. In probability theory, the lifetimes of certain devices (e.g. certain types of fuses and light bulbs) are modeled by an *Exponential Distribution*.

    (a) The probability that a device lasts more than a (time units) is $\int_a^\infty \lambda e^{-\lambda x}\,dx$ where λ is a parameter that depends on the type of device. Evaluate this integral.

    (b) The expected lifetime of the device is given by $\int_0^\infty x\lambda e^{-\lambda x}\,dx$. Evaluate this integral.

    (c) What is the probability that a given device lasts more than the expected lifetime for such devices?

48. For $n > 0$, the gamma function is defined by $\Gamma(n) = \displaystyle\int_0^\infty x^{n-1}e^{-x}\,dx$.

    (a) Show that $\Gamma(1) = 1$.

    (b) Show that $\Gamma(n+1) = n\Gamma(n)$ for $n > 1$.

    (c) Conclude that $\Gamma(n+1) = n!$ for integers n such that $n \geq 1$.

    (d) Show that this converges for $0 < n < 1$.

8.7 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute an antiderivative of the integrand. The second case is when we actually do not know the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions and their inverses. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,\mathrm{d}x$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,\mathrm{d}x, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,\mathrm{d}x, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,\mathrm{d}x,$$

as pictured in Figure 8.7.1.

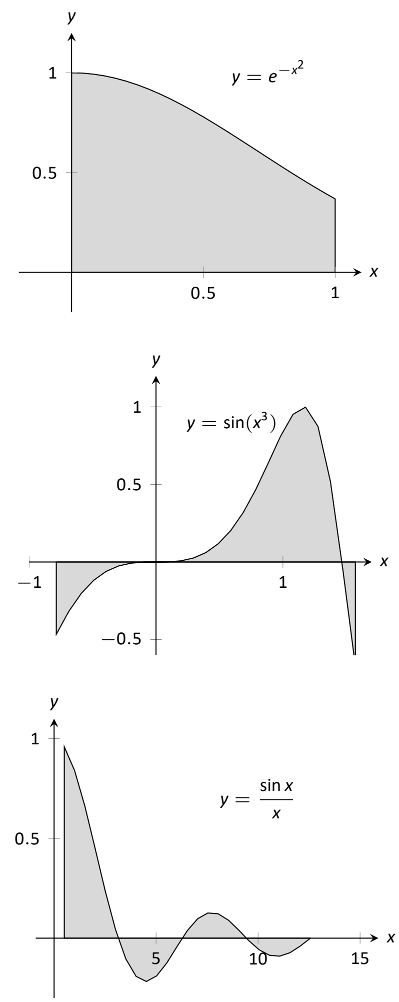


Figure 8.7.1: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

The Left and Right Hand Rule Methods

In Section 5.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a,b]$. We wish to approximate $\int_a^b f(x)\,\mathrm{d}x$. We partition $[a,b]$ into n

Notes:

equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these subintervals are labeled as

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_i = a + i\Delta x,\ \ldots,\ x_n = b.$$

Section 5.3 showed that to use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_{i-1})\Delta x$ and to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_i)\Delta x$. We review the use of these rules in the context of examples.

Example 8.7.1 Approximating definite integrals with rectangles

Approximate $\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution We begin by partitioning the interval $[0,1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,\ \text{and } x_5 = 1.$$

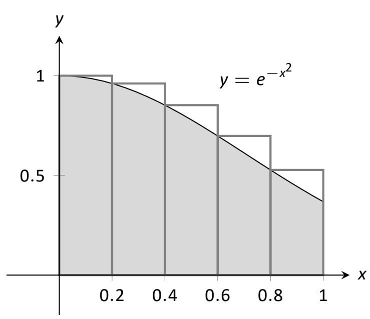
Using the Left Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_{i-1})\Delta x &= \big(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)\big)\Delta x\\ &= \big(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)\big)\Delta x\\ &\approx \big(1 + 0.961 + 0.852 + 0.698 + 0.527\big)(0.2)\\ &\approx 0.808.\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\big)\Delta x\\ &= \big(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)\big)\Delta x\\ &\approx \big(0.961 + 0.852 + 0.698 + 0.527 + 0.368\big)(0.2)\\ &\approx 0.681.\end{aligned}$$

Figure 8.7.2 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did in



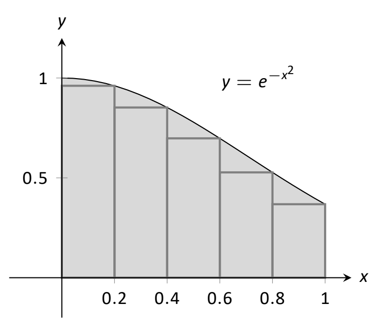


Figure 8.7.2: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 8.7.1 using (top) the left hand rule and (bottom) the right hand rule.

Notes:

Section 5.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808 + 0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

x_i	Exact	Approx.	$\sin(x_i^3)$
x_0	$-\pi/4$	-0.785	-0.466
x_1	$-7\pi/40$	-0.550	-0.165
x_2	$-\pi/10$	-0.314	-0.031
x_3	$-\pi/40$	-0.0785	0
x_4	$\pi/20$	0.157	0.004
x_5	$\pi/8$	0.393	0.061
x_6	$\pi/5$	0.628	0.246
x_7	$11\pi/40$	0.864	0.601
x_8	$7\pi/20$	1.10	0.971
x_9	$17\pi/40$	1.34	0.690
x_{10}	$\pi/2$	1.57	-0.670

Figure 8.7.3: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 8.7.2.

Example 8.7.2 Approximating definite integrals with rectangles

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

Solution We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 8.7.3, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

$$\text{Left Hand Rule: } \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.91)(0.236) = 0.451.$$

$$\text{Right Hand Rule: } \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.71)(0.236) = 0.404.$$

The average of the Left and Right Hand Rules is 0.4275. The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 8.7.4. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

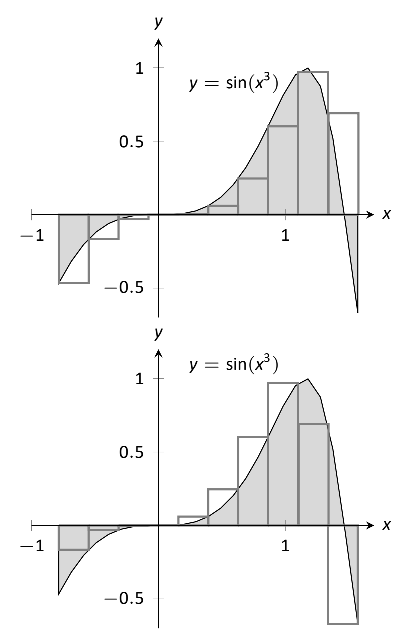


Figure 8.7.4: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 8.7.2 using (top) the left hand rule and (bottom) the right hand rule.

Notes:

The Trapezoidal Rule

In Example 8.7.1 we approximated the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 8.7.2 showed the rectangles used in the Left and Right Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 8.7.5, we show the region under $f(x) = e^{-x^2}$ on $[0,1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lie on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area.)

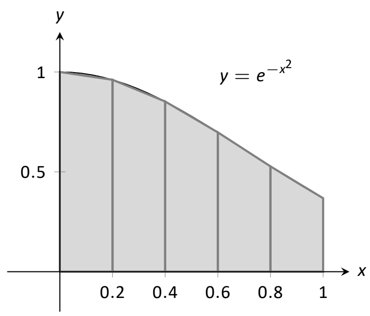


Figure 8.7.5: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

Watch the video:
The Trapezoid Rule for Approximating Integrals at
`https://youtu.be/8z6JRFvjkpc`

The formula for the area of a trapezoid is given in Figure 8.7.6. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

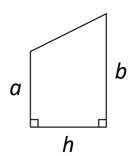


Figure 8.7.6: The area of a trapezoid is $\frac{a+b}{2}h$.

Example 8.7.3 Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution To compute the areas of the 5 trapezoids in Figure 8.7.5, it will again be useful to create a table of values as shown in Figure 8.7.7.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 8.7.7: A table of values of e^{-x^2}.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

Notes:

The sum of the areas of all 5 trapezoids is:

$$\begin{aligned}\frac{1+0.961}{2}(0.2)+\frac{0.961+0.852}{2}(0.2)+\frac{0.852+0.698}{2}(0.2)\\ +\frac{0.698+0.527}{2}(0.2)+\frac{0.527+0.368}{2}(0.2)=0.7445.\end{aligned}$$

We approximate $\int_0^1 e^{-x^2}\,dx \approx 0.7445$.

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+\\ (0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1+2(0.961+0.852+0.698+0.527)+0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, where a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_0, x_1, ..., x_n, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n}\frac{f(x_{i-1})+f(x_i)}{2}\Delta x\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_{i-1})+f(x_i)\big)\\ &= \frac{\Delta x}{2}\left[f(x_0)+2\sum_{i=1}^{n-1}f(x_i)+f(x_n)\right].\end{aligned}$$

Notes:

Example 8.7.4 Using the Trapezoidal Rule

Revisit Example 8.7.2 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

Solution We refer back to Figure 8.7.3 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}
&\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \\
&\qquad \approx \frac{0.236}{2}\Big[-0.466 + 2\Big(-0.165 + (-0.031) + \cdots + 0.69\Big) + (-0.67)\Big] \\
&\qquad = 0.4275.
\end{aligned}$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

We will also show that the Trapezoidal Rule makes using the Midpoint Rule obsolete as well. With much more work, it will turn out that the Midpoint Rule has only a marginal gain in accuracy. But we will include it in our results for the sake of completeness.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By

Notes:

approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710–1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x-value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x-values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. An exercise will ask you to show that

$$\int_{x_1}^{x_3} f(x)\,dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \qquad (8.7.1)$$

Consider Figure 8.7.8. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \approx 3.$$

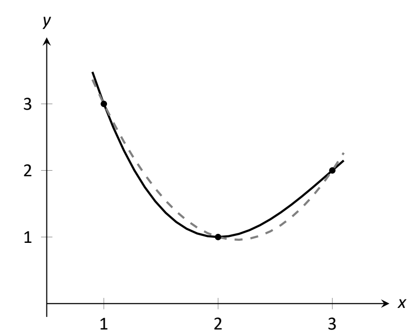


Figure 8.7.8: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b - a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (8.7.1) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\,dx \approx$$
$$\frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Notes:

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, ..., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

Example 8.7.5 Using Simpson's Rule

Approximate $\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution We begin by making a table of values as we have in the past, as shown in Figure 8.7.9(a). Simpson's Rule states that

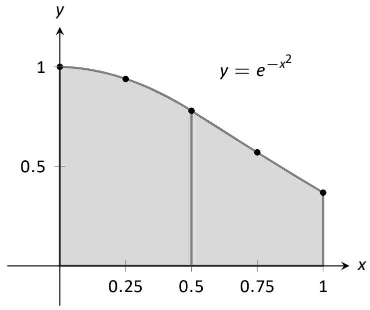
$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 8.7.1 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5.

Figure 8.7.9(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 8.7.9: A table of values to approximate $\int_0^1 e^{-x^2}\,dx$ in Example 8.7.5, along with a graph of the function.

Example 8.7.6 Using Simpson's Rule

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution Figure 8.7.10(a) shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

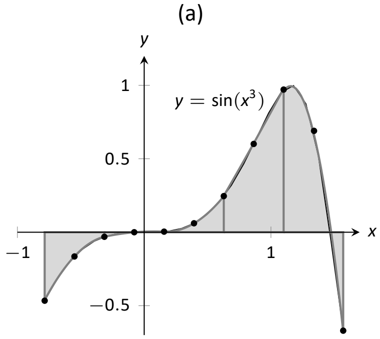
Simpson's Rule states that

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx \frac{0.236}{3}\Big[&(-0.466) + 4(-0.165) + 2(-0.031) + \cdots \\ &\cdots + 2(0.971) + 4(0.69) + (-0.67)\Big]\\ = 0.4701&\end{aligned}$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within 0.01 of the correct value. The graph in Figure 8.7.10(b) shows how closely the parabolas match the shape of the graph.

x_i	$\sin(x_i^3)$
−0.785	−0.466
−0.550	−0.165
−0.314	−0.031
−0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	−0.670

(a)



(b)

Figure 8.7.10: A table of values to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 8.7.6, along with a graph of the function.

Notes:

Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 8.7.1 Numerical Integration

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$. Set $x_0 = a$, $x_1 = a + \Delta x$, ..., $x_i = a + i\Delta x$, $x_n = b$. Then $\int_a^b f(x)\,dx$ can be approximated by:

Left Hand Rule: $\Delta x\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$.

Right Hand Rule: $\Delta x\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$.

Midpoint Rule: $\Delta x\Big[f\big(\frac{x_0+x_1}{2}\big) + f\big(\frac{x_1+x_2}{2}\big) + \cdots + f\big(\frac{x_{n-1}+x_n}{2}\big)\Big]$.

Trapezoidal Rule: $\frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$.

Simpson's Rule: $\frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 8.7.1, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more

Notes:

subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 8.7.1 states what these bounds are.

Theorem 8.7.1 Error Bounds in Numerical Integration
Suppose that K_m is an upper bound on $\left|f^{(m)}(x)\right|$ on $[a,b]$. Then a bound for the error of the numerical method of integration is given by:

Method	Error Bound
Left/Right Hand Rule	$\dfrac{K_1(b-a)^2}{2n}$
Midpoint Rule	$\dfrac{K_2(b-a)^3}{24n^2}$
Trapezoidal Rule	$\dfrac{K_2(b-a)^3}{12n^2}$
Simpson's Rule	$\dfrac{K_4(b-a)^5}{180n^4}$

There are some key things to note about this theorem.

1. The larger the interval, the larger the error. This should make sense intuitively.
2. The error shrinks as more subintervals are used (i.e., as n gets larger).
3. When n doubles, the Left and Right Hand Rules double in accuracy, the Midpoint and Trapezoidal Rules quadruple in accuracy, and Simpson's Rule is 16 times more accurate.
4. The error in Simpson's Rule has a term relating to the 4^{th} derivative of f. Consider a cubic polynomial: its 4^{th} derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 — Simpson's Rule computes the exact answer!

We revisit Examples 8.7.3 and 8.7.5 and compute the error bounds using Theorem 8.7.1 in the following example.

Notes:

Example 8.7.7 Computing error bounds

Find the error bounds when approximating $\int_0^1 e^{-x^2}\,dx$ using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.

Solution **Trapezoidal Rule with $n = 5$:**

We start by computing the 2[nd] derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 8.7.11 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem 8.7.1.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 8.7.3 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \leq \int_0^1 e^{-x^2}\,dx \leq 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 8.7.1.

Simpson's Rule with $n = 4$:

We start by computing the 4[th] derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

Figure 8.7.12 shows a graph of $f^{(4)}(x)$ on $[0, 1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem 8.7.1.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 8.7.5 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \leq \int_0^1 e^{-x^2}\,dx \leq 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem 8.7.1.

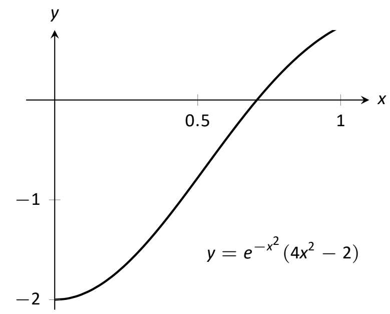


Figure 8.7.11: Graphing $f''(x)$ in Example 8.7.7 to help establish error bounds.

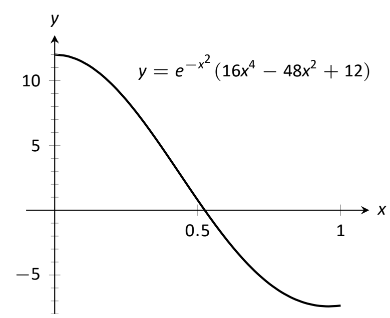


Figure 8.7.12: Graphing $f^{(4)}(x)$ in Example 8.7.7 to help establish error bounds.

Notes:

We have seen that, for $\int_0^1 e^{-x^2}\,dx$, Simpson's Rule with 4 subintervals is far more accurate than the Trapezoidal Rule with 5 subintervals. We now find how many intervals we would need to match that accuracy.

Example 8.7.8 Finding a number subintervals

Find the number of subintervals necessary to estimate $\int_0^1 e^{-x^2}\,dx$ to within 0.00026 using the Trapezoidal Rule.

Solution We can again use that $f''(x)$ is bounded by 2, so that

$$E_T = \frac{(1-0)^3}{12\cdot n^2}\cdot 2 = \frac{1}{6n^2}.$$

In order for this to be at most 0.00026, we need to have

$$n \geq \frac{1}{\sqrt{6\cdot 0.00026}} \approx 25.3.$$

Therefore, we will need at least 26 subintervals in order to have as much accuracy as Simpson's Rule with 4 subintervals.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Example 8.7.9 Approximating distance traveled

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 8.7.13. Approximate the distance they traveled.

Solution Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\int_a^b v(t)\,dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

Time (min)	Speed (mph)
0	0
0.5	25
1	22
1.5	19
2	39
2.5	0
3	43
3.5	59
4	54
4.5	51
5	43
5.5	35
6	40
6.5	43
7	30
7.5	0
8	0
8.5	28
9	40
9.5	42
10	40
10.5	39
11	40
11.5	23
12	0

Figure 8.7.13: Speed data collected at 30 second intervals for Example 8.7.9.

Notes:

We need to compute $\Delta x = (b-a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or 1/5 of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \quad \frac{\Delta x}{3} = \frac{1}{360}.$$

Thus the distance traveled is approximately:

$$\begin{aligned}\int_0^{0.2} v(t)\,dt &\approx \frac{1}{360}\Big[f(x_1) + 4f(x_2) + 2f(x_3) + \cdots + 4f(x_n) + f(x_{n+1})\Big]\\ &= \frac{1}{360}\Big[0 + 4\cdot 25 + 2\cdot 22 + \cdots + 2\cdot 40 + 4\cdot 23 + 0\Big]\\ &\approx 6.2167 \text{ miles.}\end{aligned}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

Notes:

Exercises 8.7

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating antiderivatives.
2. What are the two basic situations where approximating the value of a definite integral is necessary?
3. Why are the Left and Right Hand Rules rarely used?
4. Why is the Midpoint Rule rarely used?

Problems

In Exercises 5–12, a definite integral is given.

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) Find the exact value of the integral.

5. $\int_{-1}^{1} x^2 \, dx$
6. $\int_{0}^{10} 5x \, dx$
7. $\int_{0}^{\pi} \sin x \, dx$
8. $\int_{0}^{4} \sqrt{x} \, dx$
9. $\int_{0}^{3} (x^3 + 2x^2 - 5x + 7) \, dx$
10. $\int_{0}^{1} x^4 \, dx$
11. $\int_{0}^{2\pi} \cos x \, dx$
12. $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$

In Exercises 13–20, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

13. $\int_{0}^{1} \cos\left(x^2\right) dx$
14. $\int_{-1}^{1} e^{x^2} \, dx$
15. $\int_{0}^{5} \sqrt{x^2 + 1} \, dx$
16. $\int_{0}^{\pi} x \sin x \, dx$
17. $\int_{0}^{\pi/2} \sqrt{\cos x} \, dx$
18. $\int_{1}^{4} \ln x \, dx$
19. $\int_{-1}^{1} \frac{1}{\sin x + 2} \, dx$
20. $\int_{0}^{6} \frac{1}{\sin x + 2} \, dx$

In Exercises 21–24, find n such that the error in approximating the given definite integral is less than 0.0001 when using:

(a) the Trapezoidal Rule

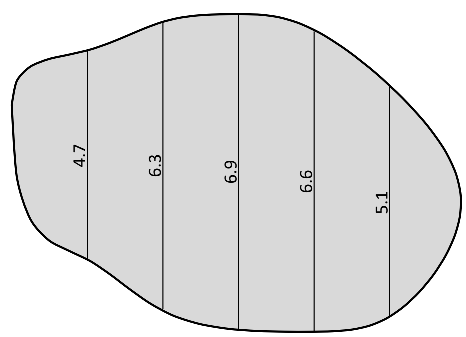
(b) Simpson's Rule

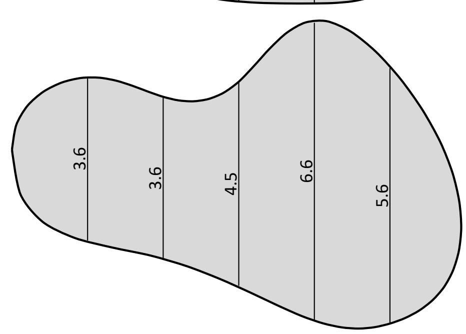
21. $\int_{0}^{\pi} \sin x \, dx$
22. $\int_{1}^{4} \frac{1}{\sqrt{x}} \, dx$
23. $\int_{0}^{\pi} \cos\left(x^2\right) dx$
24. $\int_{0}^{5} x^4 \, dx$

In Exercises 25–26, a region is given. Find the area of the region using Simpson's Rule:

(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of yards, taken in 100 yd increments.

25. 

26. 

27. Let f be the quadratic function that goes through the points (x_1, y_1), $(x_1 + \Delta x, y_2)$ and $(x_1 + 2\Delta x, y_3)$. Show that $\int_{x_1}^{x_1 + 2\Delta x} f(x) \, dx = \frac{\Delta x}{3}(y_1 + 4y_2 + y_3)$.

9: Sequences and Series

This chapter introduces **sequences** and **series**, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

9.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers $2, 4, 6, 8, \ldots$, form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ (it could also be described by $n^4 - 10n^3 + 35n^2 - 48n + 24$, to give one of infinitely many other options). To find the 10^{th} term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Notes:

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers $1, 2, 3, \ldots$

Definition 9.1.1 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1), a(2), \ldots$, which are usually denoted with subscripts as $a_1, a_2, \ldots$.

A sequence $a(n)$ is often denoted as $\{a_n\}$.

Factorial: The expression $3!$ refers to the number $3 \cdot 2 \cdot 1 = 6$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number.

We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

Watch the video:
Sequences — Examples showing convergence or divergence at
`https://youtu.be/9K1xx6wfN-U`

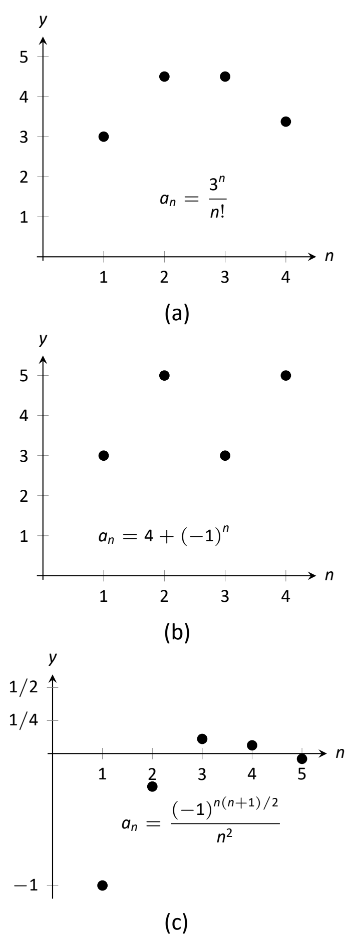


Figure 9.1.1: Plotting sequences in Example 9.1.1.

Example 9.1.1 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3^n}{n!}\right\}$ 2. $\{a_n\} = \{4 + (-1)^n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^{n(n+1)/2}}{n^2}\right\}$

SOLUTION

1. $a_1 = \dfrac{3^1}{1!} = 3;\quad a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};\quad a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};\quad a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$
   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 9.1.1(a).

2. $a_1 = 4 + (-1)^1 = 3;\quad a_2 = 4 + (-1)^2 = 5;$
   $a_3 = 4 + (-1)^3 = 3;\quad a_4 = 4 + (-1)^4 = 5.$
   Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 9.1.1(b).

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1;\quad a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4}$
   $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9}\quad a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16};$

Notes:

$$a_5 = \frac{(-1)^{5(6)/2}}{5^2} = -\frac{1}{25}.$$

We gave one extra term to begin to show the pattern of signs is "$-$, $-$, $+$, $+$, $-$, $-$, …", due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 9.1.1(c).

Example 9.1.2 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $2, 5, 8, 11, 14, \ldots$
2. $2, -5, 10, -17, 26, -37, \ldots$
3. $1, 1, 2, 6, 24, 120, 720, \ldots$
4. $\frac{5}{2}, \frac{5}{2}, \frac{15}{8}, \frac{5}{4}, \frac{25}{32}, \ldots$

Solution We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

   After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are 0!, 1!, 2!, 3!, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the 0! term. Instead, we shift by 1, and write $a_n = (n-1)!$.

Notes:

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavor is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Definition 9.1.2 Limit of a Sequence, Convergent, Divergent
Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is** L, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε-δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 9.1.1 Limit of a Sequence
Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

If $\lim\limits_{x\to\infty} f(x) = L$, then $\lim\limits_{n\to\infty} a_n = L$.

Notes:

Theorem 9.1.1 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences (and even if the theorem doesn't apply, we can still use ideas from that chapter to prove similar theorems for sequences). Note two things *not* stated by the theorem:

1. If $\lim\limits_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim\limits_{x\to\infty} f(x)$ does not exist.

   However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim\limits_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim\limits_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

Example 9.1.3 Determining convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2 - 2n + 1}{n^2 - 1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

Solution

1. Using Key Idea 1.5.2, we can state that $\lim\limits_{x\to\infty} \dfrac{3x^2 - 2x + 1}{x^2 - 1000} = 3$. (We could have also directly applied L'Hôpital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 9.1.2 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

2. The limit $\lim\limits_{x\to\infty} \cos x$ does not exist, as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). This means that we cannot apply Theorem 9.1.1.

   The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 9.1.2 (b), where the sequence is plotted, shows that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave.

   Based on the graph, we suspect that $\lim\limits_{n\to\infty} a_n$ does not exist, but we have not decisively proven it yet.

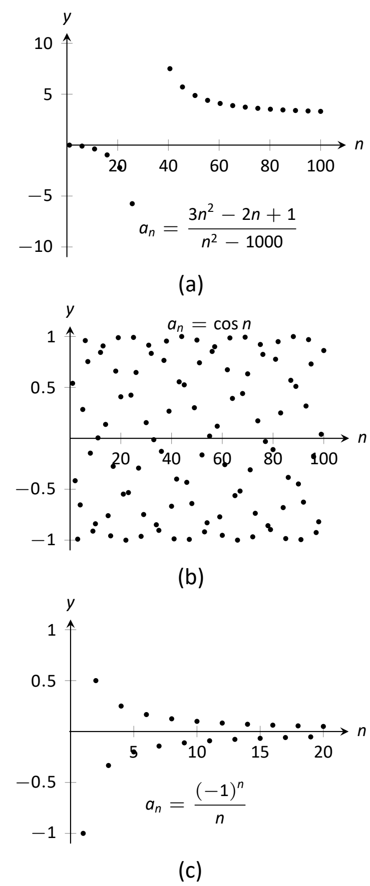


Figure 9.1.2: Scatter plots of the sequences in Example 9.1.3.

Notes:

3. We cannot actually apply Theorem 9.1.1 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.) So for now we say that we cannot determine the limit. (But we will be able to very soon.) By looking at the plot in Figure 9.1.2 (c), we would like to conclude that the sequence converges to 0. That is true, but at this point we are unable to decisively say so.

It seems that $\{(-1)^n/n\}$ converges to 0 but we lack the formal tool to prove it. The following theorem gives us that tool.

Theorem 9.1.2 Absolute Value Theorem
Let $\{a_n\}$ be a sequence. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Proof
We know $-|a_n| \leq a_n \leq |a_n|$ and $\lim\limits_{n\to\infty} (-|a_n|) = -\lim\limits_{n\to\infty} |a_n| = 0$. Thus by the Squeeze Theorem, $\lim\limits_{n\to\infty} a_n = 0$. □

Example 9.1.4 Determining the convergence / divergence of a sequence
Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ 2. $\{a_n\} = \left\{\dfrac{(-1)^n(n+1)}{n}\right\}$

Solution

1. This appeared in Example 9.1.3. We want to apply Theorem 9.1.2, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0.\end{aligned}$$

Since this limit is 0, we can apply Theorem 9.1.2 and state that $\lim\limits_{n\to\infty} a_n = 0$.

Notes:

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim_{x\to\infty} \dfrac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 9.1.2:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= 1.\end{aligned}$$

We have concluded that when we ignore the sign, the sequence approaches 1. This means we cannot apply Theorem 9.1.2; it states the the limit must be 0 in order to conclude anything.

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 9.1.3.

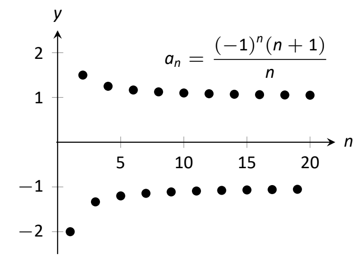


Figure 9.1.3: A plot of a sequence in Example 9.1.4, part 2.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 9.1.3 Properties of the Limits of Sequences
Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$, where L and K are real numbers, and let c be a real number.

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim_{n\to\infty} (a_n / b_n) = L/K$, $K \neq 0$
4. $\lim_{n\to\infty} (c \cdot a_n) = c \cdot L$

Example 9.1.5 Applying properties of limits of sequences
Let the following limits be given:

- $\lim_{n\to\infty} a_n = 0$;
- $\lim_{n\to\infty} b_n = e$; and
- $\lim_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim_{n\to\infty} (a_n + b_n)$ 2. $\lim_{n\to\infty} (b_n \cdot c_n)$ 3. $\lim_{n\to\infty} (1000 \cdot a_n)$

Notes:

Solution We will use Theorem 9.1.3 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty}(a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.

2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty}(b_n \cdot c_n) = e \cdot 5 = 5e$.

3. Since $\lim\limits_{n\to\infty} a_n = 0$, we have $\lim\limits_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

Definition 9.1.3 Geometric Sequence
For a constant r, the sequence $\{r^n\}$ is known as a **geometric sequence**.

Theorem 9.1.4 Convergence of Geometric Sequences
The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. Furthermore,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \end{cases}$$

Proof
We can see from Key Idea 7.3.1 and by letting $a = r$ that

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 0 & 0 < r < 1. \end{cases}$$

We also know that $\lim\limits_{x\to\infty} 1^n = 1$ and $\lim\limits_{x\to\infty} 0^n = 0$. If $-1 < r < 0$, we know $0 < |r| < 1$ so $\lim\limits_{x\to\infty} |r^n| = \lim\limits_{x\to\infty} |r|^n = 0$ and thus by Theorem 9.1.2, $\lim\limits_{x\to\infty} r^n = 0$. If $r \leq -1$, $\lim\limits_{x\to\infty} r^n$ does not exist. Therefore, the sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r. □

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Notes:

Definition 9.1.4 Bounded and Unbounded Sequences
A sequence $\{a_n\}$ is said to be **bounded** if there exist real numbers m and M such that $m < a_n < M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

Example 9.1.6 Determining boundedness of sequences
Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\dfrac{1}{n}\right\}$ 2. $\{a_n\} = \{2^n\}$

SOLUTION

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 9.1.4(a) illustrates this.
2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 9.1.4(b) illustrates this.

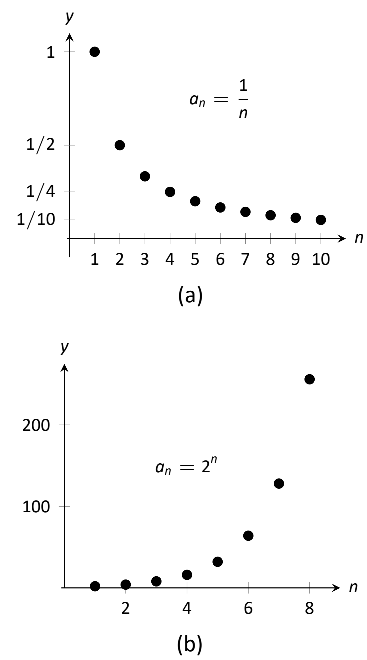


Figure 9.1.4: A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 9.1.6.

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near 0, so we could find some sort of bound on these terms (using Definition 9.1.2, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic suggests that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Notes:

Theorem 9.1.5 Convergent Sequences are Bounded
Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

Note: Keep in mind what Theorem 9.1.5 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

In Example 7.5.3 part 1, we found that $\lim_{x\to\infty}(1+1/x)^x = e$. If we consider the sequence $\{b_n\} = \{(1+1/n)^n\}$, we see that $\lim_{n\to\infty} b_n = e$. Even though it may be difficult to intuitively grasp the behavior of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 9.1.6 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 9.1.5 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.

A similar statement holds for *strictly decreasing.*

Example 9.1.7 Determining monotonicity
Determine the monotonicity of the following sequences.

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

Solution In each of the following, we will examine $a_{n+1}-a_n$. If $a_{n+1}-a_n \geq 0$, we conclude that $a_n \leq a_{n+1}$ and hence the sequence is increasing. If

Notes:

$a_{n+1} - a_n \leq 0$, we conclude that $a_n \geq a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

1.
$$\begin{aligned} a_{n+1} - a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n) - (n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1} - a_n < 0$ for all n, we conclude that the sequence is decreasing.

2.
$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1) - (n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n(n+3)}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

Since $a_{n+1} - a_n > 0$ for all n, we conclude the sequence is increasing.

3. We can clearly see in Figure 9.1.7, where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

$$\begin{aligned} a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{n^2 + 2n - 8}{n^2 - 8n + 17} - \frac{n^2 - 9}{n^2 - 10n + 26} \\ &= \frac{(n^2 + 2n - 8)(n^2 - 10n + 26) - (n^2 - 9)(n^2 - 8n + 17)}{(n^2 - 8n + 17)(n^2 - 10n + 26)} \\ &= \frac{-10n^2 + 60n - 55}{(n^2 - 8n + 17)(n^2 - 10n + 26)}. \end{aligned}$$

We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. Using the quadratic formula, we can determine that $-10n^2 + 60n -$

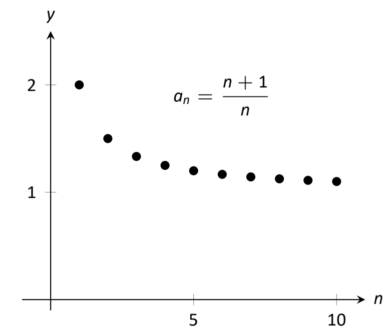


Figure 9.1.5: A plot of $\{a_n\} = \{\frac{n+1}{n}\}$ in Example 9.1.7(a).

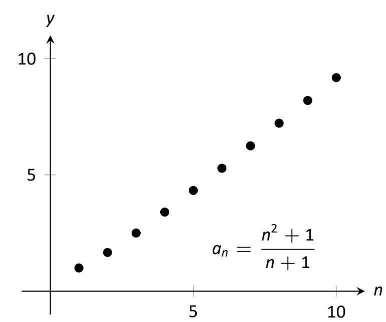


Figure 9.1.6: A plot of $\{a_n\} = \{\frac{n^2+1}{n+1}\}$ in Example 9.1.7(b).

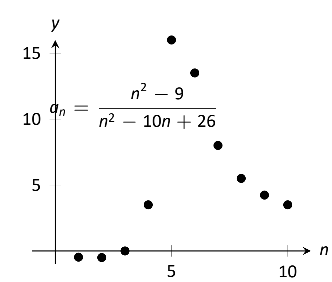


Figure 9.1.7: A plot of $\{a_n\} = \{\frac{n^2-9}{n^2-10n+26}\}$ in Example 9.1.7(c).

Notes:

$55 = 0$ when $n \approx 1.13, 4.87$. So for $n < 1.13$, the sequence is decreasing. Since we are only dealing with the natural numbers, this means that $a_1 > a_2$.

Between 1.13 and 4.87, i.e., for $n = 2$, 3 and 4, we have that $a_{n+1} > a_n$ and the sequence is increasing. (That is, when $n = 2$, 3 and 4, the numerator $-10n^2 + 60n - 55$ from the fraction above is > 0.)

When $n > 4.87$, i.e, for $n \geq 5$, we have that $-10n^2+60n-55 < 0$, hence $a_{n+1} - a_n < 0$, so the sequence is decreasing.

In short, the sequence is simply not monotonic. However, it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 9.1.8 shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. Instead of looking at $a_{n+1} - a_n$, this time we'll look at a_n/a_{n+1}:

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{n^2}{n!}\frac{(n+1)!}{(n+1)^2}\\ &= \frac{n^2}{n+1}\\ &= n-1+\frac{1}{n+1}\end{aligned}$$

When $n = 1$, the above expression is < 1; for $n \geq 2$, the above expression is > 1. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

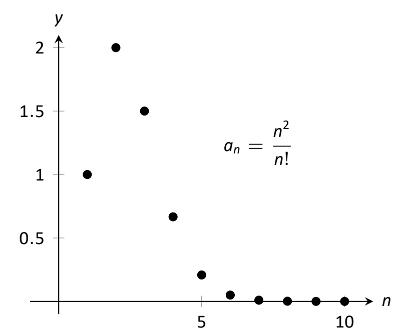


Figure 9.1.8: A plot of $\{a_n\} = \{n^2/n!\}$ in Example 9.1.7(d).

Knowing that a sequence is monotonic can be useful. In particular, if we know that a sequence is bounded and monotonic, we can conclude it converges. Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Theorem 9.1.6 Bounded Monotonic Sequences are Convergent
Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists.

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by

Notes:

0). Therefore we can conclude by Theorem 9.1.6 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

Convergence of a sequence does not depend on the first N terms of a sequence. For example, we could adapt the sequence of the previous paragraph to be

$$1,\ 10,\ 100,\ 1000,\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \frac{1}{9},\ \frac{1}{10},\ldots$$

Because we only changed three of the first 4 terms, we have not affected whether the sequence converges or diverges.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences and their formulae. (As of this writing, there are 348,000 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Notes:

Exercises 9.1

Terms and Concepts

1. Use your own words to define a *sequence.*
2. The domain of a sequence is the ________ numbers.
3. Use your own words to describe the *range* of a sequence.
4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5–8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{ \dfrac{4^n}{(n+1)!} \right\}$
6. $\{b_n\} = \left\{ \left(-\dfrac{3}{2}\right)^n \right\}$
7. $\{c_n\} = \left\{ -\dfrac{n^{n+1}}{n+2} \right\}$
8. $\{d_n\} = \left\{ \dfrac{1}{\sqrt{5}} \left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n \right) \right\}$

In Exercises 9–12, determine the n^{th} term of the given sequence.

9. $4, 7, 10, 13, 16, \ldots$
10. $3,\ -\dfrac{3}{2},\ \dfrac{3}{4},\ -\dfrac{3}{8},\ \ldots$
11. $10,\ 20,\ 40,\ 80,\ 160,\ \ldots$
12. $1, 1,\ \dfrac{1}{2},\ \dfrac{1}{6},\ \dfrac{1}{24},\ \dfrac{1}{120},\ \ldots$

In Exercises 13–16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{ \dfrac{2^n - 20}{2^n} \right\}; \qquad \lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{ \left(1 + \dfrac{2}{n}\right)^n \right\}; \qquad \lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}; \qquad \lim\limits_{n\to\infty} c_n = 0$

13. $\{d_n\} = \left\{ \dfrac{2^n - 20}{7 \cdot 2^n} \right\}$
14. $\{d_n\} = \{3b_n - a_n\}$
15. $\{d_n\} = \left\{ \sin(3/n) \left(1 + \dfrac{2}{n}\right)^n \right\}$
16. $\{d_n\} = \left\{ \left(1 + \dfrac{2}{n}\right)^{2n} \right\}$

In Exercises 17–38, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{ (-1)^n \dfrac{n}{n+1} \right\}$
18. $\{a_n\} = \left\{ \dfrac{4n^2 - n + 5}{3n^2 + 1} \right\}$
19. $\{a_n\} = \left\{ \dfrac{4^n}{5^n} \right\}$
20. $\{a_n\} = \left\{ \dfrac{(n-3)!}{(n+1)!} \right\}$
21. $\{a_n\} = \left\{ \dfrac{n-1}{n} - \dfrac{n}{n-1} \right\},\ n \geq 2$
22. $\{a_n\} = \left\{ \dfrac{6^{n+3}}{8^n} \right\}$
23. $\{a_n\} = \{\ln(n)\}$
24. $\{a_n\} = \left\{ \dfrac{3n}{\sqrt{n^2+1}} \right\}$
25. $\{a_n\} = \left\{ \left(1 + \dfrac{1}{n}\right)^n \right\}$
26. $\{a_n\} = \left\{ \dfrac{(2n+1)!}{(2n-1)!} \right\}$
27. $\{a_n\} = \left\{ 5 - \dfrac{1}{n} \right\}$
28. $\{a_n\} = \left\{ \dfrac{(-1)^{n+1}}{n} \right\}$
29. $\{a_n\} = \left\{ \dfrac{1.1^n}{n} \right\}$
30. $\{a_n\} = \left\{ \dfrac{2n}{n+1} \right\}$
31. $\{a_n\} = \left\{ (-1)^n \dfrac{n^2}{2^n - 1} \right\}$
32. $\{a_n\} = \left\{ 2 + \dfrac{9^n}{8^n} \right\}$
33. $\{a_n\} = \left\{ \dfrac{(n-1)!}{(n+1)!} \right\}$
34. $\{a_n\} = \{\ln(3n+2) - \ln n\}$
35. $\{a_n\} = \{\ln(2n^2 + 3n + 1) - \ln(n^2 + 1)\}$
36. $\{a_n\} = \left\{ n \sin\left(\dfrac{1}{n}\right) \right\}$
37. $\{a_n\} = \left\{ \dfrac{e^n + e^{-n}}{e^{2n} - 1} \right\}$
38. $\{a_n\} = \left\{ \dfrac{\ln n}{\ln 2n} \right\}$

In Exercises 39–42, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

39. $\{a_n\} = \{\sin n\}$
40. $\{a_n\} = \left\{ (-1)^n \dfrac{3n-1}{n} \right\}$
41. $\{a_n\} = \left\{ \dfrac{3n^2 - 1}{n} \right\}$
42. $\{a_n\} = \{2^n - n!\}$

In Exercises 43–48, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

43. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

44. $\{a_n\} = \left\{\dfrac{n^2-6n+9}{n}\right\}$

45. $\{a_n\} = \left\{(-1)^n \dfrac{1}{n^3}\right\}$

46. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

47. $\{a_n\} = \left\{\cos\left(\dfrac{n\pi}{2}\right)\right\}$

48. $\{a_n\} = \{ne^{-n}\}$

49. Prove Theorem 9.1.2; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

50. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

    (b) Give an example where $a_n < b_n$ for all n but $L = K$.

51. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

52. In this section, we have focused on sequences defined by a formula, $a_n = f(n)$. Another common way to define a sequence is by a recurrence relation, an equation that relates a_n to previous terms in the sequence. For example, let $a_n = ra_{n-1}$ for $n = 1, 2, 3, \ldots$, with the initial condition $a_0 = A$. (Starting the sequence with $n = 0$ is common for recurrence relations.)

    (a) Write out the first few terms of the above sequence, and find a pattern that lets you define a_n by a function $f(n)$ that does not explicitly include prior terms of the sequence.

    (b) Many recurrence relations yield sequences whose terms cannot be expressed by a simple function $f(n)$, and in fact such sequences can exhibit very unusual behavior. Read the Wikipedia article on the Logistic Map for a simple example that can exhibit a wide variety of behavior. (There is nothing to turn in for this part, unless your instructor gives you further directions.)

53. Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_n = \sqrt{2 + a_{n-1}}$ for all $n > 1$. So $a_1 = 1$, $a_2 = \sqrt{3}$, $a_3 = \sqrt{2+\sqrt{3}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{3}}}$,....

    (a) Show that $a_n < 2$ for all $n \geq 1$.

    (b) Show that $\{a_n\}$ is increasing.

    (c) Conclude that $\{a_n\}$ converges.

    (d) Determine the limit of $\{a_n\}$.

9.2 Infinite Series

Given the sequence $\{a_n\} = \{1/2^n\} = 1/2,\ 1/4,\ 1/8,\ \ldots$, consider the following sums:

$$\begin{array}{rcccl} a_1 & = & 1/2 & = & 1/2 \\ a_1 + a_2 & = & 1/2 + 1/4 & = & 3/4 \\ a_1 + a_2 + a_3 & = & 1/2 + 1/4 + 1/8 & = & 7/8 \\ a_1 + a_2 + a_3 + a_4 & = & 1/2 + 1/4 + 1/8 + 1/16 & = & 15/16 \end{array}$$

Later, we will be able to show that

$$a_1 + a_2 + a_3 + \cdots + a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

Let S_n be the sum of the first n terms of the sequence $\{1/2^n\}$. From the above, we see that $S_1 = 1/2$, $S_2 = 3/4$, and that $S_n = 1 - 1/2^n$.

Now consider the following limit: $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \big(1 - 1/2^n\big) = 1$. This limit can be interpreted as saying something amazing: *the sum of* all *the terms of the sequence* $\{1/2^n\}$ *is 1.*

This example illustrates some interesting concepts that we explore in this section. We begin this exploration with some definitions.

Definition 9.2.1 Infinite Series, n^{th} Partial Sums, Convergence, Divergence

Let $\{a_n\}$ be a sequence.

1. The sum $\displaystyle\sum_{n=1}^{\infty} a_n$ is an **infinite series** (or, simply **series**).
2. Let $S_n = \displaystyle\sum_{i=1}^{n} a_i$; the sequence $\{S_n\}$ is the sequence of n^{th} **partial sums** of $\{a_n\}$.
3. If the sequence $\{S_n\}$ converges to L, we say the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges** to L, and we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$.
4. If the sequence $\{S_n\}$ diverges, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **diverges**.

Using our new terminology, we can state that the series $\displaystyle\sum_{n=1}^{\infty} 1/2^n$ converges,

Notes:

and $\displaystyle\sum_{n=1}^{\infty} 1/2^n = 1$.

Watch the video:
Finding a Formula for a Partial Sum of a Telescoping Series at
https://youtu.be/cyoiIBs7kIg

We will explore a variety of series in this section. We start with two series that diverge, showing how we might discern divergence.

Example 9.2.1 Showing series diverge

1. Let $\{a_n\} = \{n^2\}$. Show $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.
2. Let $\{b_n\} = \{(-1)^{n+1}\}$. Show $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Solution

1. Consider S_n, the n^{th} partial sum.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= 1^2 + 2^2 + 3^2 \cdots + n^2 \\ &= \frac{n(n+1)(2n+1)}{6}. \qquad \text{by Theorem 5.3.1} \end{aligned}$$

Since $\lim_{n\to\infty} S_n = \infty$, we conclude that the series $\displaystyle\sum_{n=1}^{\infty} n^2$ diverges. It is instructive to write $\displaystyle\sum_{n=1}^{\infty} n^2 = \infty$ for this tells us *how* the series diverges: it grows without bound.

A scatter plot of the sequences $\{a_n\}$ and $\{S_n\}$ is given in Figure 9.2.1. The terms of $\{a_n\}$ are growing, so the terms of the partial sums $\{S_n\}$ are growing even faster, illustrating that the series diverges.

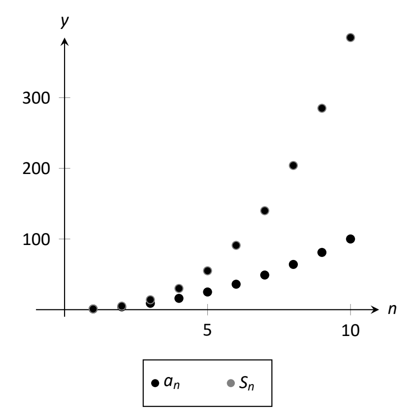


Figure 9.2.1: Scatter plots relating to the series of Example 9.2.1 part 1.

Notes:

2. The sequence $\{b_n\}$ starts with $1, -1, 1, -1, \ldots$. Consider some of the partial sums S_n of $\{b_n\}$:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 0 \\ S_3 &= 1 \\ S_4 &= 0 \end{aligned}$$

This pattern repeats; we find that $S_n = \begin{cases} 1 & n \text{ is odd} \\ 0 & n \text{ is even.} \end{cases}$ As $\{S_n\}$ oscillates, repeating $1, 0, 1, 0, \ldots$, we conclude that $\lim_{n\to\infty} S_n$ does not exist, hence $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges.

A scatter plot of the sequence $\{b_n\}$ and the partial sums $\{S_n\}$ is given in Figure 9.2.2. When n is odd, $b_n = S_n$ so the marks for b_n are drawn oversized to show they coincide.

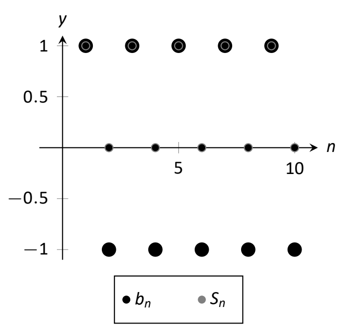


Figure 9.2.2: Scatter plots relating to the series of Example 9.2.1 part 2.

While it is important to recognize when a series diverges, we are generally more interested in the series that converge. In this section we will demonstrate a few general techniques for determining convergence; later sections will delve deeper into this topic.

Geometric Series

One important type of series is a *geometric series*.

Definition 9.2.2 Geometric Series
A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

Note that the index starts at $n = 0$. If the index starts at $n = 1$ we have $\sum_{n=1}^{\infty} ar^{n-1}$.

Notes:

We started this section with a geometric series, although we dropped the first term of 1. One reason geometric series are important is that they have nice convergence properties.

Theorem 9.2.1 Convergence of Geometric Series

Consider the geometric series $\displaystyle\sum_{n=0}^{\infty} ar^n$ where $a \neq 0$.

1. If $r \neq 1$, the n^{th} partial sum is: $S_n = \displaystyle\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$.
2. The series converges if, and only if, $|r| < 1$. When $|r| < 1$,
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Proof

If $r = 1$, then $S_n = a+a+a+\cdots+a = na$. Since $\displaystyle\lim_{n\to\infty} S_n = \pm\infty$, the geometric series diverges.
If $r \neq 1$, we have
$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$
Multiply each term by r and we have
$$rS_n = ar + ar^2 + ar^3 \cdots + ar^n.$$
Subtract these two equations and solve for S_n.
$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

From Theorem 9.1.4, we know that if $-1 < r < 1$, then $\displaystyle\lim_{n\to\infty} r^n = 0$ so
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n = \frac{a}{1-r}.$$
So when $|r| < 1$ the geometric series converges and its sum is $\dfrac{a}{1-r}$.

If either $r \leq -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by Theorem 9.1.4. Thus $\displaystyle\lim_{n\to\infty} S_n$ does not exist, so the geometric series diverges if $r \leq -1$ or $r > 1$.

$\square$

Notes:

According to Theorem 9.2.1, the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

converges as $r = 1/2$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$. This concurs with our introductory example; while there we got a sum of 1, we skipped the first term of 1.

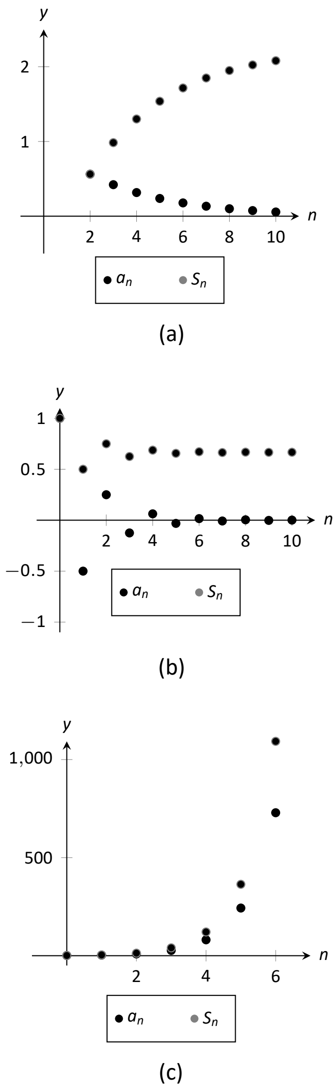


Figure 9.2.3: Scatter plots relating to the series in Example 9.2.2.

Example 9.2.2 Exploring geometric series

Check the convergence of the following series. If the series converges, find its sum.

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n$ 2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$ 3. $\displaystyle\sum_{n=0}^{\infty} 3^n$

Solution

1. Since $r = 3/4 < 1$, this series converges. By Theorem 9.2.1, we have that

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-3/4} = 4.$$

However, note the subscript of the summation in the given series: we are to start with $n = 2$. Therefore we subtract off the first two terms, giving:

$$\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = 4 - 1 - \frac{3}{4} = \frac{9}{4}.$$

This is illustrated in Figure 9.2.3(a).

2. Since $|r| = 1/2 < 1$, this series converges, and by Theorem 9.2.1,

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{2}{3}.$$

The partial sums of this series are plotted in Figure 9.2.3(b). Note how the partial sums are not purely increasing as some of the terms of the sequence $\{(-1/2)^n\}$ are negative.

3. Since $r > 1$, the series diverges. (This makes "common sense"; we expect the sum

$$1 + 3 + 9 + 27 + 81 + 243 + \cdots$$

to diverge.) This is illustrated in Figure 9.2.3(c).

Notes:

Later sections will provide tests by which we can determine whether or not a given series converges. This, in general, is much easier than determining *what* a given series converges to. There are many cases, though, where the sum can be determined.

Example 9.2.3 Telescoping series

Evaluate the sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution It will help to write down some of the first few partial sums of this series.

$$\begin{aligned}
S_1 &= \frac{1}{1}-\frac{1}{2} && = 1-\frac{1}{2}\\
S_2 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) && = 1-\frac{1}{3}\\
S_3 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) && = 1-\frac{1}{4}\\
S_4 &= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) && = 1-\frac{1}{5}
\end{aligned}$$

Note how most of the terms in each partial sum subtract out. In general, we see that $S_n = 1-\frac{1}{n+1}$. This means that the sequence $\{S_n\}$ converges, as

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\tfrac{1}{n+1}\right) = 1,$$

and so we conclude that $\displaystyle\sum_{n=1}^{\infty}\left(\tfrac{1}{n}-\tfrac{1}{n+1}\right) = 1$. Partial sums of the series are plotted in Figure 9.2.4.

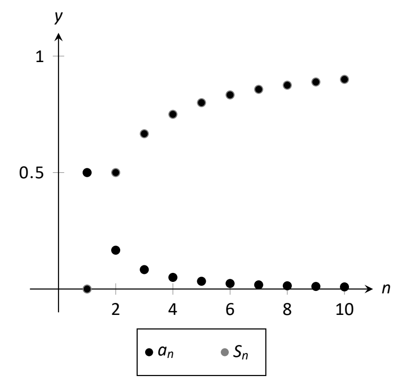


Figure 9.2.4: Scatter plots relating to the series of Example 9.2.3.

The series in Example 9.2.3 is an example of a **telescoping series**. Informally, a telescoping series is one in which the partial sums reduce to just a fixed number of terms. The partial sum S_n did not contain n terms, but rather just two: 1 and $1/(n+1)$.

When possible, seek a way to write an explicit formula for the n^{th} partial sum S_n. This makes evaluating the limit $\lim\limits_{n\to\infty} S_n$ much more approachable. We do so in the next example.

Notes:

Example 9.2.4 Evaluating series

Evaluate each of the following infinite series.

$$1.\quad \sum_{n=1}^{\infty} \frac{2}{n^2+2n} \qquad\qquad 2.\quad \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

SOLUTION

1. We can decompose the fraction $2/(n^2+2n)$ as

$$\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}.$$

(See Section 8.4, Partial Fraction Decomposition, to recall how this is done, if necessary.)

Expressing the terms of $\{S_n\}$ is now more instructive:

$$\begin{aligned}
S_1 &= 1 - \tfrac{1}{3} && = 1 - \tfrac{1}{3}\\
S_2 &= (1 - \tfrac{1}{3}) + (\tfrac{1}{2} - \tfrac{1}{4}) && = 1 + \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4}\\
S_3 &= (1 - \tfrac{1}{3}) + (\tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{3} - \tfrac{1}{5}) && = 1 + \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5}\\
S_4 &= (1 - \tfrac{1}{3}) + (\tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{3} - \tfrac{1}{5}) + (\tfrac{1}{4} - \tfrac{1}{6}) && = 1 + \tfrac{1}{2} - \tfrac{1}{5} - \tfrac{1}{6}\\
S_5 &= (1 - \tfrac{1}{3}) + (\tfrac{1}{2} - \tfrac{1}{4}) + (\tfrac{1}{3} - \tfrac{1}{5}) + (\tfrac{1}{4} - \tfrac{1}{6}) + (\tfrac{1}{5} - \tfrac{1}{7}) && = 1 + \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{7}
\end{aligned}$$

We again have a telescoping series. In each partial sum, most of the terms pair up to add to zero and we obtain the formula $S_n = 1 + \dfrac{1}{2} - \dfrac{1}{n+1} - \dfrac{1}{n+2}$. Taking limits allows us to determine the convergence of the series:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{3}{2},$$

so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \frac{3}{2}$. This is illustrated in Figure 9.2.5.

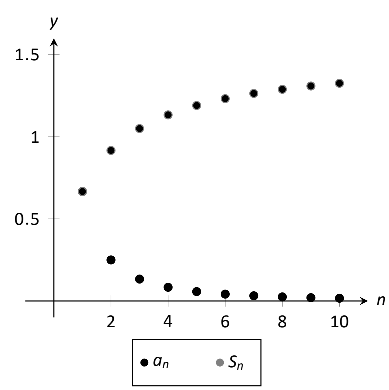


Figure 9.2.5: Scatter plots relating to the series of Example 9.2.4 part 1.

2. We begin by writing the first few partial sums of the series:

$$\begin{aligned}
S_1 &= \ln(2)\\
S_2 &= \ln(2) + \ln\left(\frac{3}{2}\right)\\
S_3 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\\
S_4 &= \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right)
\end{aligned}$$

Notes:

At first, this does not seem helpful, but recall the logarithmic identity: $\ln x + \ln y = \ln(xy)$. Applying this to S_4 gives:

$$S_4 = \ln(2) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \ln\left(\frac{5}{4}\right) = \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\right) = \ln(5).$$

We must generalize this for S_n.

$$\begin{aligned} S_n &= \ln(2) + \ln\left(\frac{3}{2}\right) + \cdots + \ln\left(\frac{n+1}{n}\right) \\ &= \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n}{n-1}\cdot\frac{n+1}{n}\right) = \ln(n+1) \end{aligned}$$

We can conclude that $\{S_n\} = \{\ln(n+1)\}$. This sequence does not converge, as $\lim_{n\to\infty} S_n = \infty$. Therefore $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \infty$; the series diverges. Note in Figure 9.2.6 how the sequence of partial sums grows slowly; after 100 terms, it is not yet over 5. Graphically we may be fooled into thinking the series converges, but our analysis above shows that it does not.

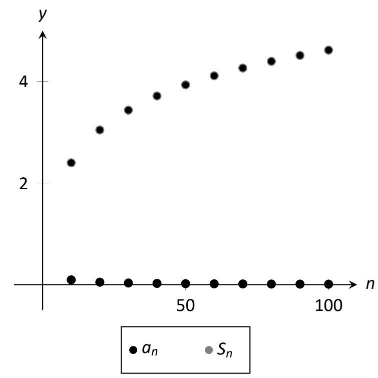


Figure 9.2.6: Scatter plots relating to the series of Example 9.2.4 part 2.

We are learning about a new mathematical object, the series. As done before, we apply "old" mathematics to this new topic.

Theorem 9.2.2 Properties of Infinite Series

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are convergent series, and that $\displaystyle\sum_{n=1}^{\infty} a_n = L$, $\displaystyle\sum_{n=1}^{\infty} b_n = K$, and c is a constant.

1. Constant Multiple Rule: $\displaystyle\sum_{n=1}^{\infty} c\cdot a_n = c\cdot\sum_{n=1}^{\infty} a_n = c\cdot L.$
2. Sum/Difference Rule: $\displaystyle\sum_{n=1}^{\infty}\left(a_n \pm b_n\right) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = L \pm K.$

Before using this theorem, we will consider the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

Notes:

Example 9.2.5 Divergence of the Harmonic Series

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution We will use a proof by contradiction here. Suppose the harmonic series converges to S. That is

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

We then have

$$\begin{aligned} S &\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{2} \quad + \frac{1}{3} \quad + \frac{1}{4} \quad + \cdots \\ &= \frac{1}{2} + S \end{aligned}$$

This gives us $S \geq \frac{1}{2} + S$ which can never be true, thus our assumption that the harmonic series converges must be false. Therefore, the harmonic series diverges.

It may take a while before one is comfortable with this statement, whose truth lies at the heart of the study of infinite series: *it is possible that the sum of an infinite list of nonzero numbers is finite.* We have seen this repeatedly in this section, yet it still may "take some getting used to."

As one contemplates the behavior of series, a few facts become clear.

1. In order to add an infinite list of nonzero numbers and get a finite result, "most" of those numbers must be "very near" 0.
2. If a series diverges, it means that the sum of an infinite list of numbers is not finite (it may approach $\pm\infty$ or it may oscillate), and:
   (a) The series will still diverge if the first term is removed.
   (b) The series will still diverge if the first 10 terms are removed.
   (c) The series will still diverge if the first $1{,}000{,}000$ terms are removed.
   (d) The series will still diverge if any finite number of terms from anywhere in the series are removed.

These concepts are very important and lie at the heart of the next two theorems.

Notes:

Theorem 9.2.3 Convergence of Sequence

If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof

Let $S_n = a_1 + a_2 + \cdots + a_n$. We have

$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ S_n &= S_{n-1} + a_n \\ a_n &= S_n - S_{n-1} \end{aligned}$$

Since $\sum_{n\to\infty} a_n$ converges, the sequence $\{S_n\}$ converges. Let $\lim_{n\to\infty} S_n = S$. As $n \to \infty$, $n-1$ also goes to ∞, so $\lim_{n\to\infty} S_{n-1} = S$. We now have

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} \\ &= S - S = 0 \end{aligned}$$

□

Theorem 9.2.4 Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

The Test for Divergence follows from Theorem 9.2.3. If the series does not diverge, it must converge and therefore $\lim_{n\to\infty} a_n = 0$.

Note that the two statements in Theorems 9.2.3 and 9.2.4 are really the same. In order to converge, the terms of the sequence must approach 0; if they do not, the series will not converge.

Looking back, we can apply this theorem to the series in Example 9.2.1. In that example, we had $\{a_n\} = \{n^2\}$ and $\{b_n\} = \{(-1)^{n+1}\}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$$

and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (-1)^{n+1} \text{ which does not exist.}$$

Notes:

Thus by the Test for Divergence, both series will diverge.

Important! This theorem *does not state* that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges. The standard example of this is the Harmonic Series, as given in Example 9.2.5. The Harmonic Sequence, $\{1/n\}$, converges to 0; the Harmonic Series, $\sum_{n=1}^{\infty} 1/n$, diverges.

Theorem 9.2.5 Infinite Nature of Series
The convergence or divergence of a series remains unchanged by the insertion or deletion of any finite number of terms. That is:

1. A divergent series will remain divergent with the insertion or deletion of any finite number of terms.
2. A convergent series will remain convergent with the insertion or deletion of any finite number of terms. (Of course, the *sum* will likely change.)

In other words, when we are only interested in the convergence or divergence of a series, it is safe to ignore the first few billion terms.

Example 9.2.6 Removing Terms from the Harmonic Series

Consider once more the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges; that is, the partial sums $S_N = \sum_{n=1}^{N} \frac{1}{n}$ grow (very, very slowly) without bound. One might think that by removing the "large" terms of the sequence that perhaps the series will converge. This is simply not the case. For instance, the sum of the first 10 million terms of the Harmonic Series is about 16.7. Removing the first 10 million terms from the Harmonic Series changes the partial sums, effectively subtracting 16.7 from the sum. However, a sequence that is growing without bound will still grow without bound when 16.7 is subtracted from it.

The equation below illustrates this. Even though we have subtracted off the first 10 million terms, this only subtracts a constant off of an expression that is still growing to infinity. Therefore, the modified series is still growing to infinity.

$$\begin{aligned}\sum_{n=10{,}000{,}001}^{\infty} \frac{1}{n} = \lim_{N\to\infty} \sum_{n=10{,}000{,}001}^{N} \frac{1}{n} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{10{,}000{,}001} \frac{1}{n}\\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - 16.7 = \infty.\end{aligned}$$

Notes:

This section introduced us to series and defined a few special types of series whose convergence properties are well known. We know when a geometric series converges or diverges. Most series that we encounter are not one of these types, but we are still interested in knowing whether or not they converge. The next three sections introduce tests that help us determine whether or not a given series converges.

Notes:

Exercises 9.2

Terms and Concepts

1. Use your own words to describe how sequences and series are related.
2. Use your own words to define a *partial sum*.
3. Given a series $\sum_{n=1}^{\infty} a_n$, describe the two sequences related to the series that are important.
4. Use your own words to explain what a geometric series is.
5. T/F: If $\{a_n\}$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
6. T/F: If $\{a_n\}$ converges to 0, then $\sum_{n=0}^{\infty} a_n$ converges.

Problems

In Exercises 7–14, a series $\sum_{n=1}^{\infty} a_n$ is given.

(a) Give the first 5 partial sums of the series.

(b) Give a graph of the first 5 terms of a_n and S_n on the same axes.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \cos(\pi n)$

10. $\displaystyle\sum_{n=1}^{\infty} n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$

14. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

In Exercises 15–32, state whether the given series converges or diverges and provide justification for your conclusion.

15. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{5^n}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{5^n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

19. $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}$

20. $\displaystyle\sum_{n=0}^{\infty} \left(\ln(4n+2) - \ln(7n+5)\right)$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n!} + \frac{1}{n}\right)$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 1}{2^{n+1}}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$

26. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{3}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n + 2^n}{6^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n + 5^n}{6^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{5}{4^n}\right)$

In Exercises 33–48, a series is given.

(a) Find a formula for S_n, the n^{th} partial sum of the series.

(b) Determine whether the series converges or diverges. If it converges, state what it converges to.

33. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{4^n}$

34. $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

35. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

36. $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n}$

37. $\sum_{n=1}^{\infty} e^{-n}$

38. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots$

39. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

40. $\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$

41. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$

42. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

43. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

44. $\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \frac{1}{4\cdot 7} + \cdots$

45. $2 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{9}\right) + \left(\frac{1}{8} + \frac{1}{27}\right) + \cdots$

46. $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

47. $\sum_{n=0}^{\infty} (\sin 1)^n$

48. $\sum_{n=1}^{\infty} \left(\frac{2}{n(n+2)} + \frac{5}{4^n}\right)$

In Exercises 49–52, find the values of x for which the series converges.

49. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

50. $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n}$

51. $\sum_{n=1}^{\infty} \frac{4^n}{x^n}$

52. $\sum_{n=1}^{\infty} (x+2)^n$

In Exercises 53–58, use Theorem 9.2.4 to show the given series diverges.

53. $\sum_{n=1}^{\infty} \frac{3n^2}{n(n+2)}$

54. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

55. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

56. $\sum_{n=1}^{\infty} \frac{5^n - n^5}{5^n + n^5}$

57. $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$

58. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

59. Show the series $\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)}$ diverges.

60. Rewrite $0.1212\overline{12}\ldots$ as an infinite series and then express the sum as the quotient of two integers (thus showing that it is a rational number — this method can be generalized to show that every repeating decimal is rational).

61. A ball falling from a height of h meters is known to rebound to a height of rh meters, where the proportionality constant r satisfies $0 < r < 1$. Find the total distance traveled (vertically) by the ball if it is dropped initially from a height of 2 meters.

9.3 The Integral Test

Knowing whether or not a series converges is very important, especially when we discuss Power Series in Section 9.8. Theorem 9.2.1 gives criteria for when Geometric series converge and Theorem 9.2.4 gives a quick test to determine if a series diverges. There are many important series whose convergence cannot be determined by these theorems, though, so we introduce a set of tests that allow us to handle a broad range of series. We start with the Integral Test.

Integral Test

We stated in Section 9.1 that a sequence $\{a_n\}$ is a function $a(n)$ whose domain is $\mathbb{N}$, the set of natural numbers. If we can extend $a(n)$ to have the domain of $\mathbb{R}$, the real numbers, and it is both positive and decreasing on $[1, \infty)$, then the convergence of $\sum_{n=1}^{\infty} a_n$ is the same as $\int_1^{\infty} a(x)\, dx$.

Theorem 9.3.1 Integral Test

Let a sequence $\{a_n\}$ be defined by $a_n = a(n)$, where $a(n)$ is continuous, positive, and decreasing on $[1, \infty)$. Then $\sum_{n=1}^{\infty} a_n$ converges, if, and only if, $\int_1^{\infty} a(x)\, dx$ converges. In other words:

1. If $\int_1^{\infty} a(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\int_1^{\infty} a(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: Theorem 9.3.1 does not state that the integral and the summation have the same value.

Note that it is not necessary to start the series or the integral at $n = 1$. We may use any interval $[n, \infty)$ on which $a(n)$ is continuous, positive and decreasing. Also the sequence $\{a_n\}$ does not have to be strictly decreasing. It must be *ultimately decreasing* which means it is decreasing for all n larger than some number N.

We can demonstrate the truth of the Integral Test with two simple graphs. In Figure 9.3.1(a), the height of each rectangle is $a(n) = a_n$ for $n = 1, 2, \ldots,$

Notes:

and clearly the rectangles enclose more area than the area under $y = a(x)$. Therefore we can conclude that

$$\int_1^{\infty} a(x)\,\mathrm{d}x < \sum_{n=1}^{\infty} a_n. \tag{9.3.1}$$

In Figure 9.3.1(b), we draw rectangles under $y = a(x)$ with the Right-Hand rule, starting with $n = 2$. This time, the area of the rectangles is less than the area under $y = a(x)$, so $\displaystyle\sum_{n=2}^{\infty} a_n < \int_1^{\infty} a(x)\,\mathrm{d}x$. Note how this summation starts with $n = 2$; adding a_1 to both sides lets us rewrite the summation starting with $n = 1$:

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,\mathrm{d}x. \tag{9.3.2}$$

Combining Equations (9.3.1) and (9.3.2), we have

$$\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,\mathrm{d}x < a_1 + \sum_{n=1}^{\infty} a_n. \tag{9.3.3}$$

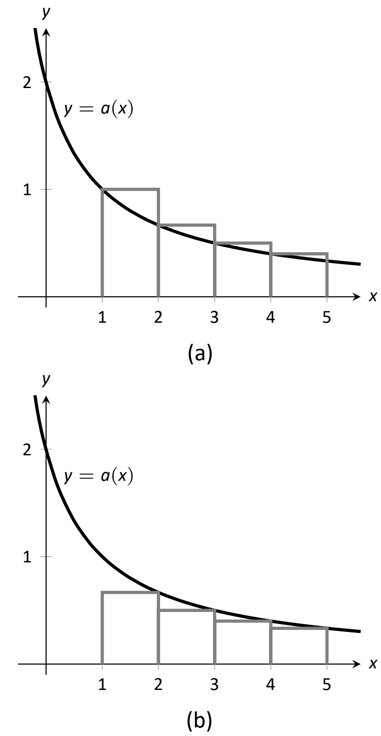


Figure 9.3.1: Illustrating the truth of the Integral Test.

From Equation (9.3.3) we can make the following two statements:

1. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, so does $\displaystyle\int_1^{\infty} a(x)\,\mathrm{d}x$

   (because $\displaystyle\sum_{n=1}^{\infty} a_n < a_1 + \int_1^{\infty} a(x)\,\mathrm{d}x$)

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, so does $\displaystyle\int_1^{\infty} a(x)\,\mathrm{d}x$

   (because $\displaystyle\int_1^{\infty} a(x)\,\mathrm{d}x < \sum_{n=1}^{\infty} a_n$.)

Therefore the series and integral either both converge or both diverge. Theorem 9.2.5 allows us to extend this theorem to series where $a(n)$ is positive and decreasing on $[b, \infty)$ for some $b > 1$.

Watch the video:
Integral Test for Series: Why It Works at
`https://youtu.be/ObiRjUFHJHo`

Notes:

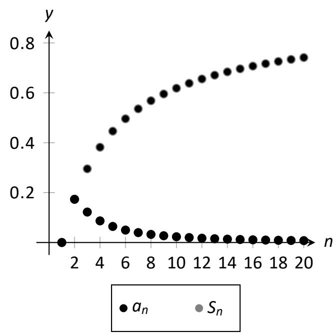


Figure 9.3.2: Plotting the sequence and series in Example 9.3.1.

Example 9.3.1 Using the Integral Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. (The terms of the sequence $\{a_n\} = \{(\ln n)/n^2\}$ and the n^th partial sums are given in Figure 9.3.2.)

Solution Figure 9.3.2 implies that $a(n) = (\ln n)/n^2$ is positive and decreasing on $[2, \infty)$. We can determine this analytically, too. We know $a(n)$ is positive as both $\ln n$ and n^2 are positive on $[2, \infty)$. To determine that $a(n)$ is decreasing, consider $a'(n) = (1 - 2\ln n)/n^3$, which is negative for $n \geq 2$. Since $a'(n)$ is negative, $a(n)$ is decreasing.

Applying the Integral Test, we test the convergence of $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$. Integrating this improper integral requires the use of Integration by Parts, with $u = \ln x$ and $dv = 1/x^2\, dx$.

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{x^2}\, dx &= \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x^2}\, dx \\
&= \lim_{t\to\infty} \left(-\frac{1}{x}\ln x\Big|_1^t + \int_1^t \frac{1}{x^2}\, dx \right) \\
&= \lim_{t\to\infty} \left(\left[-\frac{1}{x}\ln x - \frac{1}{x} \right]_1^t \right) \\
&= \lim_{t\to\infty} \left(1 - \frac{1}{t} - \frac{\ln t}{t} \right). \quad \text{Apply L'Hôpital's Rule:} \\
&= 1 - 0 - \lim_{t\to\infty} \frac{1}{t} \\
&= 1
\end{aligned}$$

Since $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$.

p-Series

Another important type of series is the *p-series*.

Notes:

Definition 9.3.1 *p*-**Series, General** *p*-**Series**

1. A *p*-**series** is a series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$.

2. A **general** *p*-**series** is a series of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$,

   where $a > 0$, b is a real number, and $an + b \neq 0$ for all n.

Like geometric series, one of the nice things about *p*-series is that they have easy to determine convergence properties.

Theorem 9.3.2 **Convergence of General** *p*-**Series**
Assume a and b are real numbers and $an + b \neq 0$ for all n.
A general *p*-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ will converge if, and only if, $p > 1$.

Proof
Consider the integral $\displaystyle\int_1^{\infty} \frac{1}{(ax+b)^p}\, dx$; assuming $p \neq 1$,

$$\begin{aligned}\int_1^{\infty} \frac{1}{(ax+b)^p}\, dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{(ax+b)^p}\, dx \\ &= \lim_{t\to\infty} \frac{1}{a(1-p)}(ax+b)^{1-p}\Big|_1^t \\ &= \lim_{t\to\infty} \frac{1}{a(1-p)}\big((at+b)^{1-p} - (a+b)^{1-p}\big).\end{aligned}$$

This limit converges if and only if, $p > 1$. It is easy to show that the integral also diverges in the case of $p = 1$. (This result is similar to the work preceding Key Idea 8.6.1.)

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if, and only if, $p > 1$. □

Notes:

Example 9.3.2 Determining convergence of series

Determine the convergence of the following series.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
5. $\displaystyle\sum_{n=11}^{\infty} \frac{1}{(\frac{1}{2}n-5)^3}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution

1. This is a p-series with $p = 1$. By Theorem 9.3.2, this series diverges.

   This series is a famous series, called the *Harmonic Series*, so named because of its relationship to *harmonics* in the study of music and sound.

2. This is a p-series with $p = 2$. By Theorem 9.3.2, it converges. Note that the theorem does not give a formula by which we can determine *what* the series converges to; we just know it converges. A famous, unexpected result is that this series converges to $\pi^2/6$.

3. This is a p-series with $p = 1/2$; the theorem states that it diverges.

4. This is not a p-series; the definition does not allow for alternating signs. Therefore we cannot apply Theorem 9.3.2. We will consider this series again in Section 9.5. (Another famous result states that this series, the *Alternating Harmonic Series*, converges to $-\ln 2$.)

5. This is a general p-series with $p = 3$, therefore it converges.

6. This is not a p-series, but a geometric series with $r = 1/2$. It converges.

In the next section we consider two more convergence tests, both comparison tests. That is, we determine the convergence of one series by comparing it to another series with known convergence.

Notes:

Exercises 9.3

Terms and Concepts

1. In order to apply the Integral Test to a sequence $\{a_n\}$, the function $a(n) = a_n$ must be ______, ______ and ______.
2. T/F: The Integral Test can be used to determine the sum of a convergent series.

Problems

In Exercises 3–10, use the Integral Test to determine the convergence of the given series.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$
8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

In Exercises 11–14, find the value(s) of p for which the series is convergent.

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$
12. $\displaystyle\sum_{n=1}^{\infty} n(1+n^2)^p$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$
14. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\ln n)]^p}$
15. It can be shown that $\int_0^1 x^{-x}\, dx = \sum_{n=1}^{\infty} n^{-n}$. Use the Integral Test to show that the series is convergent, and hence conclude that the integral is convergent.
    Hint: remember, you only need to show that the integral in the Integral Test is convergent–you do not need to be able to evaluate it.

9.4 Comparison Tests

In this section we will be comparing a given series with series that we know either converge or diverge.

Theorem 9.4.1 Direct Comparison Test
Let $\{a_n\}$ and $\{b_n\}$ be positive sequences where $a_n \leq b_n$ for all $n \geq N$, for some $N \geq 1$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

Note: A sequence $\{a_n\}$ is a **positive sequence** if $a_n > 0$ for all n.

Because of Theorem 9.2.5, any theorem that relies on a positive sequence still holds true when $a_n > 0$ for all but a finite number of values of n.

Proof
First consider the partial sums of each series.

$$S_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad T_n = \sum_{i=1}^{n} b_i$$

Since both series have positive terms we know that

$$S_n \leq S_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = S_{n+1}$$

and

$$T_n \leq T_n + b_{n+1} = \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = T_{n+1}$$

Therefore, both of the sequences of partial sums,$\{S_n\}$ and $\{T_n\}$, are increasing.

For $n \geq N$, we're now going to split each series into two parts:

$$S = \sum_{i=1}^{N-1} a_i \qquad\qquad T = \sum_{i=1}^{N-1} b_i$$

$$\bar{S}_n = \sum_{i=N}^{n} a_i \qquad\qquad \bar{T}_n = \sum_{i=N}^{n} b_i.$$

Notes:

This means that $S_n = S + \bar{S}_n$ and $T_n = T + \bar{T}_n$. Also, because $a_n \leq b_n$ for all $n \geq N$, we must have $\bar{S}_n \leq \bar{T}_n$.

For the first part of the theorem, assume that $\displaystyle\sum_{n=1}^{\infty} b_n$ converges. Since $b_n \geq 0$ we know that

$$T_n = \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{\infty} b_i$$

From above we know that $\bar{S}_n \leq \bar{T}_n$ for all $n \geq N$ so we also have

$$S_n = S + \bar{S}_n \leq S + \bar{T}_n = S + T_n - T = S - T + \sum_{i=1}^{\infty} b_i$$

Because $\displaystyle\sum_{i=1}^{\infty} b_i$ converges it must have a finite value and $\{S_n\}$ is bounded above. We also showed that $\{S_n\}$ is increasing so by Theorem 9.1.6 we know $\{S_n\}$ converges and so $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

For the second part, assume that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. Because $a_n \geq 0$ we must have $\displaystyle\lim_{n\to\infty} S_n = \infty$. We also know that for all $n \geq N$, $\bar{S}_n \leq \bar{T}_n$. This means that

$$\begin{aligned}\lim_{n\to\infty} T_n &= \lim_{n\to\infty} (T + \bar{T}_n) \\ &\geq T + \lim_{n\to\infty} \bar{S}_n = T + \lim_{n\to\infty} (S_n - S) = T - S + \lim_{n\to\infty} S_n = \infty.\end{aligned}$$

Therefore, $\{T_n\}$ is a divergent sequence and so $\displaystyle\sum_{i=1}^{\infty} b_n$ diverges. ☐

Watch the video:
Direct Comparison Test / Limit Comparison Test
for Series — Basic Info at
`https://youtu.be/LAHKu3B-1zE`

Notes:

Example 9.4.1 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$.

Solution This series is neither a geometric or p-series, but seems related. We predict it will converge, so we look for a series with larger terms that converges. (Note too that the Integral Test seems difficult to apply here.)

Since $3^n < 3^n + n^2$, $\dfrac{1}{3^n} > \dfrac{1}{3^n + n^2}$ for all $n \geq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series; by Theorem 9.4.1, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + n^2}$ converges.

Example 9.4.2 Applying the Direct Comparison Test

Determine the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 1}$.

Solution We know the Harmonic Series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and it seems that the given series is closely related to it, hence we predict it will diverge.

We have $\dfrac{n^3}{n^4 - 1} > \dfrac{n^3}{n^4} = \dfrac{1}{n}$ for all $n \geq 2$.

The Harmonic Series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, diverges, so we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 - 1}$ diverges as well.

The concept of direct comparison is powerful and often relatively easy to apply. Practice helps one develop the necessary intuition to quickly pick a proper series with which to compare. However, it is easy to construct a series for which it is difficult to apply the Direct Comparison Test.

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4 + 1}$. It is very similar to the divergent series given in Example 9.4.2. We suspect that it also diverges, as $\dfrac{1}{n} \approx \dfrac{n^3}{n^4 + 1}$ for large n. However, the inequality that we naturally want to use "goes the wrong way": since $\dfrac{n^3}{n^4 + 1} < \dfrac{n^3}{n^4} = \dfrac{1}{n}$ for all $n \geq 1$. The given series has terms *less than* the terms of a divergent series, and we cannot conclude anything from this.

Fortunately, we can apply another test to the given series to determine its convergence.

Notes:

Limit Comparison Test

Theorem 9.4.2 Limit Comparison Test
Let $\{a_n\}$ and $\{b_n\}$ be positive sequences.

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$, where L is a positive real number, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.
2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, then if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Proof

1. We have $0 < L < \infty$ so we can find two positive numbers, m and M such that $m < L < M$. Because $L = \lim_{n\to\infty} \dfrac{a_n}{b_n}$ we know that for large enough n the quotient $\frac{a_n}{b_n}$ must be close to L. So there must be a positive integer N such that if $n > N$ we also have $m < \dfrac{a_n}{b_n} < M$. Multiply by b_n and we have $mb_n < a_n < Mb_n$ for $n > N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} mb_n$. Also since $mb_n < a_n$ for sufficiently large n, by the Comparison Test $\sum_{n=1}^{\infty} a_n$ also diverges.
   Similarly, if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} Mb_n$. Since $a_n < Mb_n$ for sufficiently large n, by the Comparison Test $\sum_{n=1}^{\infty} a_n$ also converges.
2. Since $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, there is a number $N > 0$ such that

Notes:

$$\left|\frac{a_n}{b_n} - 0\right| < 1 \text{ for all } n > N$$
$$a_n < b_n \text{ since } a_n \text{ and } b_n \text{ are positive}$$

Now since $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges by the Comparison Test.

3. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, there is a number $N > 0$ such that

$$\frac{a_n}{b_n} > 1 \text{ for all } n > N$$
$$a_n > b_n \text{ for all } n > N$$

Now since $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ diverges by the Comparison Test. □

Theorem 9.4.2 is most useful when the convergence of the series from $\{b_n\}$ is known and we are trying to determine the convergence of the series from $\{a_n\}$.

We use the Limit Comparison Test in the next example to examine the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ which motivated this new test.

Example 9.4.3 Applying the Limit Comparison Test

Determine the convergence of $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ using the Limit Comparison Test.

Solution We compare the terms of $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ to the terms of the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{n^3/(n^4+1)}{1/n} = \lim_{n\to\infty} \frac{n^4}{n^4+1} = \lim_{n\to\infty} \frac{1}{1+1/n^4}$$
$$= 1.$$

Since the Harmonic Series diverges, we conclude that $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ diverges as well.

Notes:

Example 9.4.4 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$

Solution This series is similar to the one in Example 9.4.1, but now we are considering "$3^n - n^2$" instead of "$3^n + n^2$." This difference makes applying the Direct Comparison Test difficult.

Instead, we use the Limit Comparison Test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/(3^n - n^2)}{1/3^n} &= \lim_{n\to\infty} \frac{3^n}{3^n - n^2} \\
&\overset{\text{by LHR}}{=} \lim_{n\to\infty} \frac{\ln 3 \cdot 3^n}{\ln 3 \cdot 3^n - 2n} \\
&\overset{\text{by LHR}}{=} \lim_{n\to\infty} \frac{(\ln 3)^2 3^n}{(\ln 3)^2 3^n - 2} \\
&\overset{\text{by LHR}}{=} \lim_{n\to\infty} \frac{(\ln 3)^3 3^n}{(\ln 3)^3 3^n} = 1.
\end{aligned}$$

We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is a convergent geometric series, hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - n^2}$ converges as well.

As mentioned before, practice helps one develop the intuition to quickly choose a series with which to compare. A general rule of thumb is to pick a series based on the dominant term in the expression of $\{a_n\}$. It is also helpful to note that factorials dominate increasing exponentials, which dominate algebraic functions (e.g., polynomials), which dominate logarithms. In the previous example, the dominant term of $\dfrac{1}{3^n - n^2}$ was 3^n, so we compared the series to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$. It is hard to apply the Limit Comparison Test to series containing factorials, though, as we have not learned how to apply L'Hôpital's Rule to $n!$.

Example 9.4.5 Applying the Limit Comparison Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + 3}{n^2 - n + 1}$.

Solution We naïvely attempt to apply the rule of thumb given above and note that the dominant term in the expression of the series is $1/n^2$. Knowing

Notes:

that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we attempt to apply the Limit Comparison Test:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^2} &= \lim_{n\to\infty} \frac{n^2(\sqrt{n}+3)}{n^2-n+1} \\ &= \infty \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Theorem 9.4.2 part (3) only applies when $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges; in our case, it converges. Ultimately, our test has not revealed anything about the convergence of our series.

The problem is that we chose a poor series with which to compare. Since the numerator and denominator of the terms of the series are both algebraic functions, we should have compared our series to the dominant term of the numerator divided by the dominant term of the denominator.

The dominant term of the numerator is $n^{1/2}$ and the dominant term of the denominator is n^2. Thus we should compare the terms of the given series to $n^{1/2}/n^2 = 1/n^{3/2}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{(\sqrt{n}+3)/(n^2-n+1)}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{3/2}(\sqrt{n}+3)}{n^2-n+1} \\ &= 1 \quad \text{(Apply L'Hôpital's Rule).}\end{aligned}$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2-n+1}$ converges as well.

The tests we have encountered so far have required that we analyze series from *positive* sequences (the absolute value of the ratio and the root tests of Section 9.6 will convert the sequence into a positive sequence). The next section relaxes this restriction by considering *alternating series*, where the underlying sequence has terms that alternate between being positive and negative.

Notes:

Exercises 9.4

Terms and Concepts

1. Suppose $\sum_{n=0}^{\infty} a_n$ is convergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $0 \leq b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$?

2. Suppose $\sum_{n=0}^{\infty} a_n$ is divergent, and there are sequences $\{b_n\}$ and $\{c_n\}$ such that $0 \leq b_n \leq a_n \leq c_n$ for all n. What can be said about the series $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$?

Problems

In Exercises 3–8, use the Direct Comparison Test to determine the convergence of the given series; state what series is used for comparison.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n-5}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+n^2-n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!+n}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$

8. $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}-2}$

In Exercises 9–14, use the Limit Comparison Test to determine the convergence of the given series; state what series is used for comparison.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-3n+5}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n-n^2}$

11. $\displaystyle\sum_{n=4}^{\infty} \frac{\ln n}{n-3}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

14. $\displaystyle\sum_{n=1}^{\infty} \sin\left(1/n\right)$

In Exercises 15–24, use the Direct Comparison Test or the Limit Comparison Test to determine the convergence of the given series. State which series is used for comparison.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-5}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n-10}{n^2+10n+10}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+10}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-5}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2-1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+100}$

23. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2\ln n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2+17}$

In Exercises 25–32, determine the convergence of the given series. State the test used; more than one test may be appropriate.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+5)^3}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n+n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{10n+5}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

33. Given that $\sum_{n=1}^{\infty} a_n$ converges, state which of the following series converges, may converge, or does not converge.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$

    (b) $\displaystyle\sum_{n=1}^{\infty} a_n a_{n+1}$

    (c) $\displaystyle\sum_{n=1}^{\infty} (a_n)^2$

    (d) $\displaystyle\sum_{n=1}^{\infty} n a_n$

    (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$

34. We have shown that the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ diverges. Suppose we remove some terms by considering the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ where p_n is the n^{th} prime (so $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc). Determine if this series converges or diverges, using the fact that $p_n < 2n \ln n$ for all n sufficiently large.

9.5 Alternating Series and Absolute Convergence

The series convergence tests we have used require that the underlying sequence $\{a_n\}$ be a positive sequence. (We can relax this with Theorem 9.2.5 and state that there must be an $N > 0$ such that $a_n > 0$ for all $n > N$; that is, $\{a_n\}$ is positive for all but a finite number of values of n.)

In this section we explore series whose summation includes negative terms. We start with a very specific form of series, where the terms of the summation alternate between being positive and negative.

Definition 9.5.1 Alternating Series
Let $\{b_n\}$ be a positive sequence. An **alternating series** is a series of either the form

$$\sum_{n=1}^{\infty}(-1)^n b_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} b_n.$$

We want to think that an alternating sequence $\{a_n\}$ is related to a positive sequence $\{b_n\}$ by $a_n = (-1)^n b_n$.

Recall that the terms of Harmonic Series come from the Harmonic Sequence $\{b_n\} = \{\frac{1}{n}\}$. An important alternating series is the **Alternating Harmonic Series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Geometric Series can also be alternating series when $r < 0$. For instance, if $r = -1/2$, the geometric series is

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$$

Theorem 9.2.1 states that geometric series converge when $|r| < 1$ and gives the sum: $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. When $r = -1/2$ as above, we find

$$\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3}.$$

A powerful convergence theorem exists for other alternating series that meet a few conditions.

Notes:

Theorem 9.5.1 Alternating Series Test

Let $\{b_n\}$ be a positive, decreasing sequence where $\lim\limits_{n\to\infty} b_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^n b_n \qquad \text{and} \qquad \sum_{n=1}^{\infty}(-1)^{n+1} b_n$$

converge.

The basic idea behind Theorem 9.5.1 is illustrated in Figure 9.5.1. A positive, decreasing sequence $\{b_n\}$ is shown along with the partial sums

$$S_n = \sum_{i=1}^{n}(-1)^{i+1}b_i = b_1 - b_2 + b_3 - b_4 + \cdots + (-1)^{n+1}b_n.$$

Because $\{b_n\}$ is decreasing, the amount by which S_n bounces up and down decreases. Moreover, the odd terms of S_n form a decreasing, bounded sequence, while the even terms of S_n form an increasing, bounded sequence. Since bounded, monotonic sequences converge (see Theorem 9.1.6) and the terms of $\{b_n\}$ approach 0, we will show below that the odd and even terms of S_n converge to the same common limit L, the sum of the series.

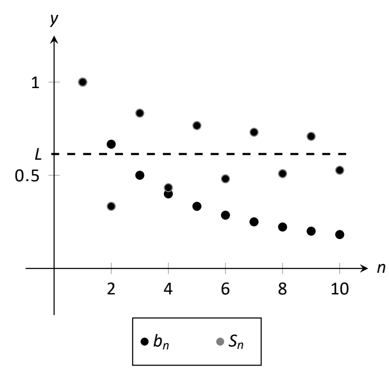


Figure 9.5.1: Illustrating convergence with the Alternating Series Test.

Proof

Because $\{b_n\}$ is a decreasing sequence, we have $b_n - b_{n+1} \geq 0$. We will consider the even and odd partial sums separately. First consider the even partial sums.

$$\begin{aligned}
S_2 &= b_1 - b_2 \geq 0 && \text{since } b_2 \leq b_1\\
S_4 &= b_1 - b_2 + b_3 - b_4 = S_2 + b_3 - b_4 \geq S_2 && \text{since } b_3 - b_4 \geq 0\\
S_6 &= S_4 + b_5 - b_6 \geq S_4 && \text{since } b_5 - b_6 \geq 0\\
&\vdots\\
S_{2n} &= S_{2n-2} + b_{2n-1} - b_{2n} \geq S_{2n-2} && \text{since } b_{2n-1} - b_{2n} \geq 0
\end{aligned}$$

We now have

$$0 \leq S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2n} \leq \cdots$$

so $\{S_{2n}\}$ is an increasing sequence. But we can also write

$$\begin{aligned}
S_{2n} &= b_1 - b_2 + b_3 - b_4 + b_5 - \cdots - b_{2n-2} + b_{2n-1} - b_{2n}\\
&= b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}
\end{aligned}$$

Each term in parentheses is positive and b_{2n} is positive so we have $S_{2n} \leq b_1$ for all n. We now have the sequence of even partial sums, $\{S_{2n}\}$, is increasing

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and bounded above so by Theorem 9.1.6 $\{S_{2n}\}$ converges. Since we know it converges, we will assume it's limit is L or

$$\lim_{n\to\infty} S_{2n} = L$$

Next we determine the limit of the sequence of odd partial sums.

$$\begin{aligned}\lim_{n\to\infty} S_{2n+1} &= \lim_{n\to\infty} (S_{2n} + b_{2n+1})\\ &= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1}\\ &= L + 0\\ &= L\end{aligned}$$

Both the even and odd partial sums converge to L so we have $\lim\limits_{n\to\infty} S_n = L$, which means the series is convergent. $\square$

Watch the video:
Alternating Series — Another Example 4 at
`https://youtu.be/a0iZvfFAMW8`

Example 9.5.1 Applying the Alternating Series Test

Determine if the Alternating Series Test applies to each of the following series.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ 2. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n}\frac{\ln n}{n}$ 3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{|\sin n|}{n^2}$

Solution

1. This is the Alternating Harmonic Series as seen previously. The underlying sequence is $\{b_n\} = \{1/n\}$, which is positive, decreasing, and approaches 0 as $n \to \infty$. Therefore we can apply the Alternating Series Test and conclude this series converges.

   While the test does not state what the series converges to, we will see later that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$.

2. The underlying sequence is $\{b_n\} = \{(\ln n)/n\}$. This is positive for $n \geq 2$ and $\displaystyle\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$ (use L'Hôpital's Rule). However, the sequence is not decreasing for all n. It is straightforward to compute $b_1 \approx$

Notes:

0.347, $b_2 \approx 0.366$, and $b_3 \approx 0.347$: the sequence is increasing for at least the first 2 terms.

We do not immediately conclude that we cannot apply the Alternating Series Test. Rather, consider the long-term behavior of $\{b_n\}$. Treating $b_n = b(n)$ as a continuous function of n defined on $[2, \infty)$, we can take its derivative:

$$b'(n) = \frac{1 - \ln n}{n^2}.$$

The derivative is negative for all $n \geq 3$ (actually, for all $n > e$), meaning $b(n) = b_n$ is decreasing on $[3, \infty)$. We can apply the Alternating Series Test to the series when we start with $n = 3$ and conclude that $\sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n}$ converges; adding the terms with $n = 2$ does not change the convergence (i.e., we apply Theorem 9.2.5).

The important lesson here is that as before, if a series fails to meet the criteria of the Alternating Series Test on only a finite number of terms, we can still apply the test.

3. The underlying sequence is $\{b_n\} = \{|\sin n| / n^2\}$. This sequence is positive and approaches 0 as $n \to \infty$. However, it is not a decreasing sequence; the value of $|\sin n|$ oscillates between 0 and 1 as $n \to \infty$. We cannot remove a finite number of terms to make $\{b_n\}$ decreasing, therefore we cannot apply the Alternating Series Test.

   Keep in mind that this does not mean we conclude the series diverges; in fact, it does converge. We are just unable to conclude this based on Theorem 9.5.1.

One of the famous results of mathematics is that the Harmonic Series, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, yet the Alternating Harmonic Series, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, converges. The notion that alternating the signs of the terms in a series can make a series converge leads us to the following definitions.

Notes:

Definition 9.5.2 Absolute and Conditional Convergence

1. A series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. A series $\sum_{n=1}^{\infty} a_n$ **converges conditionally** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Thus we say the Alternating Harmonic Series converges conditionally.

Note: In Definition 9.5.2, $\sum_{n=1}^{\infty} a_n$ is not necessarily an alternating series; it just may have some negative terms.

Example 9.5.2 Determining absolute and conditional convergence.

Determine if the following series converge absolutely, conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$
2. $\displaystyle\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$

SOLUTION

1. We can show the series

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n+3}{n^2+2n+5}\right| = \sum_{n=1}^{\infty}\frac{n+3}{n^2+2n+5}$$

diverges using the Limit Comparison Test, comparing with $1/n$.

The sequence $\{\frac{n+3}{n^2+2n+5}\}$ is monotonically decreasing, so that the series $\sum_{n=1}^{\infty}(-1)^n\frac{n+3}{n^2+2n+5}$ converges using the Alternating Series Test; we conclude it converges conditionally.

2. The series

$$\sum_{n=3}^{\infty}\left|(-1)^n\frac{3n-3}{5n-10}\right| = \sum_{n=3}^{\infty}\frac{3n-3}{5n-10}$$

diverges using the Test for Divergence, so it does not converge absolutely.

The series $\sum_{n=3}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ fails the conditions of the Alternating Series Test as $(3n-3)/(5n-10)$ does not approach 0 as $n\to\infty$. We can state

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further that this series diverges; as $n \to \infty$, the series effectively adds and subtracts $3/5$ over and over. This causes the sequence of partial sums to oscillate and not converge.

Therefore the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n-3}{5n-10}$ diverges.

Knowing that a series converges absolutely allows us to make two important statements, given in the following theorem. The first is that absolute convergence is "stronger" than regular convergence. That is, just because $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, we cannot conclude that $\displaystyle\sum_{n=1}^{\infty} |a_n|$ will converge, but knowing a series converges absolutely tells us that $\displaystyle\sum_{n=1}^{\infty} a_n$ will converge.

Theorem 9.5.2 Absolute Convergence Theorem

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series that converges absolutely.

1. $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.
2. Let $\{b_n\}$ be any rearrangement of the sequence $\{a_n\}$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

One reason this is important is that our convergence tests all require that the underlying sequence of terms be positive. By taking the absolute value of the terms of a series where not all terms are positive, we are often able to apply an appropriate test and determine absolute convergence. This, in turn, determines that the series we are given also converges.

The second statement relates to **rearrangements** of series. When dealing with a finite set of numbers, the sum of the numbers does not depend on the order which they are added. (So $1+2+3=3+1+2$.) One may be surprised to find out that when dealing with an infinite set of numbers, the same statement does not always hold true: some infinite lists of numbers may be rearranged in different orders to achieve different sums. The theorem states that the terms of

Notes:

an absolutely convergent series can be rearranged in any way without affecting the sum.

The theorem states that rearranging the terms of an absolutely convergent series does not affect its sum. This implies that perhaps the sum of a conditionally convergent series can change based on the arrangement of terms. Indeed, it can. The Riemann Rearrangement Theorem (named after Bernhard Riemann) states that any conditionally convergent series can have its terms rearranged so that the sum is any desired value or infinity.

Before we consider an example, we state the following theorem that illustrates how the alternating structure of an alternating series is a powerful tool when approximating the sum of a convergent series.

Theorem 9.5.3 The Alternating Series Approximation Theorem
Let $\{b_n\}$ be a sequence that satisfies the hypotheses of the Alternating Series Test, and let S_n and L be the n^{th} partial sum and sum, respectively, of either $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ or $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}b_n$. Then

1. $|S_n - L| < b_{n+1}$, and
2. L is between S_n and S_{n+1}.

Part 1 of Theorem 9.5.3 states that the n^{th} partial sum of a convergent alternating series will be within b_{n+1} of its total sum. Consider the alternating series we looked at before the statement of the theorem, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. Since $b_{14} = 1/14^2 \approx 0.0051$, we know that S_{13} is within 0.0051 of the total sum.

Moreover, Part 2 of the theorem states that since $S_{13} \approx 0.8252$ and $S_{14} \approx 0.8201$, we know the sum L lies between 0.8201 and 0.8252. One use of this is the knowledge that S_{14} is accurate to two places after the decimal.

Some alternating series converge slowly. In Example 9.5.1 we determined the series $\displaystyle\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converged. With $n = 1001$, we find $(\ln n)/n \approx 0.0069$, meaning that $S_{1000} \approx 0.1633$ is accurate to one, maybe two, places after the decimal. Since $S_{1001} \approx 0.1564$, we know the sum L is $0.1564 \leq L \leq 0.1633$.

Notes:

Example 9.5.3 Approximating the sums of convergent alternating series

Approximate the sum of the following series, accurate to within 0.001.

$$1.\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3} \qquad 2.\ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}.$$

Solution

1. Using Theorem 9.5.3, we want to find n where $1/n^3 \leq 0.001$:

$$\begin{aligned}\frac{1}{n^3} &\leq 0.001 = \frac{1}{1000}\\ n^3 &\geq 1000\\ n &\geq \sqrt[3]{1000}\\ n &\geq 10.\end{aligned}$$

Let L be the sum of this series. By Part 1 of the theorem, $|S_9 - L| < b_{10} = 1/1000$. We can compute $S_9 = 0.902116$, which our theorem states is within 0.001 of the total sum.

We can use Part 2 of the theorem to obtain an even more accurate result. As we know the 10th term of the series is $-1/1000$, we can easily compute $S_{10} = 0.901116$. Part 2 of the theorem states that L is between S_9 and S_{10}, so $0.901116 < L < 0.902116$.

2. We want to find n where $(\ln n)/n \leq 0.001$. We start by solving $(\ln n)/n = 0.001$ for n. This cannot be solved algebraically, so we will use Newton's Method to approximate a solution.

Let $f(x) = \ln(x)/x - 0.001$; we want to know where $f(x) = 0$. We make a guess that x must be "large," so our initial guess will be $x_1 = 1000$. Recall how Newton's Method works: given an approximate solution x_n, our next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We find $f'(x) = \big(1 - \ln(x)\big)/x^2$. This gives

$$\begin{aligned}x_2 &= 1000 - \frac{\ln(1000)/1000 - 0.001}{\big(1 - \ln(1000)\big)/1000^2}\\ &= 2000.\end{aligned}$$

Notes:

Using a computer, we find that Newton's Method seems to converge to a solution $x = 9118.01$ after 8 iterations. Taking the next integer higher, we have $n = 9119$, where $\ln(9119)/9119 = 0.000999903 < 0.001$.

Again using a computer, we find $S_{9118} = -0.160369$. Part 1 of the theorem states that this is within 0.001 of the actual sum L. Already knowing the 9,119$^{\text{th}}$ term, we can compute $S_{9119} = -0.159369$, meaning

$$-0.160369 < L < -0.159369.$$

Notice how the first series converged quite quickly, where we needed only 10 terms to reach the desired accuracy, whereas the second series took over 9,000 terms.

We now consider the Alternating Harmonic Series once more. We have stated that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}\cdots = \ln 2,$$

(see Example 9.5.1).

Consider the rearrangement where every positive term is followed by two negative terms:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12}\cdots$$

(Convince yourself that these are exactly the same numbers as appear in the Alternating Harmonic Series, just in a different order.) Now group some terms and simplify:

$$\begin{aligned}\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots &= \\ \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots &= \\ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) &= \frac{1}{2}\ln 2.\end{aligned}$$

By rearranging the terms of the series, we have arrived at a different sum. (One could *try* to argue that the Alternating Harmonic Series does not actually converge to $\ln 2$, because rearranging the terms of the series *shouldn't* change the sum. However, the Alternating Series Test proves this series converges to L, for some number L, and if the rearrangement does not change the sum, then $L = L/2$, implying $L = 0$. But the Alternating Series Approximation Theorem

Notes:

quickly shows that $L > 0$. The only conclusion is that the rearrangement *did* change the sum.) This is an incredible result.

We mentioned earlier that the Integral Test did not work well with series containing factorial terms. The next section introduces the Ratio Test, which does handle such series well. We also introduce the Root Test, which is good for series where each term is raised to a power.

Notes:

Exercises 9.5

Terms and Concepts

1. Why is $\sum_{n=1}^{\infty} \sin n$ not an alternating series?

2. A series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges when $\{a_n\}$ is ______, ______ and $\lim_{n\to\infty} a_n =$ ______.

3. Give an example of a series where $\sum_{n=0}^{\infty} a_n$ converges but $\sum_{n=0}^{\infty} |a_n|$ does not.

4. The sum of a ______ convergent series can be changed by rearranging the order of its terms.

Problems

In Exercises 5–18, an alternating series $\sum_{n=i}^{\infty} a_n$ is given.

(a) Determine if the series converges or diverges.

(b) Determine if $\sum_{n=0}^{\infty} |a_n|$ converges or diverges.

(c) If $\sum_{n=0}^{\infty} a_n$ converges, determine if the convergence is conditional or absolute.

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

6. $\sum_{n=0}^{\infty} (-e)^{-n}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{n+5}{3n-5}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^2}$

9. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{3n+5}{n^2-3n+1}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n + 1}$

11. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+3+5+\cdots+(2n-1)}$

13. $\sum_{n=1}^{\infty} \cos\left(\pi n\right)$

14. $\sum_{n=2}^{\infty} \frac{\sin\left((n+1/2)\pi\right)}{n \ln n}$

15. $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$

16. $\sum_{n=0}^{\infty} (-1)^n 2^{-n^2}$

17. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$

Let S_n be the n^{th} partial sum of a series. In Exercises 19–22, a convergent alternating series is given and a value of n. Compute S_n and S_{n+1} and use these values to find bounds on the sum of the series.

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}, \quad n=5$

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}, \quad n=4$

21. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, \quad n=6$

22. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n, \quad n=9$

In Exercises 23–26, a convergent alternating series is given along with its sum and a value of ε. Use Theorem 9.5.3 to find n such that the n^{th} partial sum of the series is within ε of the sum of the series.

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720}, \quad \varepsilon = 0.001$

24. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \quad \varepsilon = 0.0001$

25. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \quad \varepsilon = 0.001$

26. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos 1, \quad \varepsilon = 10^{-8}$

27. The partial sums in problems 23 and 25 can be used to approximate π. Using the values of n from these problems, compute the respective partial sums and then use them to approximate π. Which gives a better estimate of π?

28. The book shows a rearrangement of the Alternating Harmonic Series,

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\ldots$$

which gives the sum $\frac{1}{2}\ln 2$. Note that the terms in parentheses are positive, so if we simplified those terms we would have an alternating series. Without actually simplifying, show that the same scheme of rearranging terms so that each positive term is followed by two successive negative terms yields

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}-\frac{1}{8}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{12}-\frac{1}{16}+$$
$$\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{20}-\frac{1}{24}+\ldots$$

and then show that this new rearrangement of the Alternating Harmonic Series has the sum $\frac{1}{4}\ln 2$.
Hint: move each right parenthesis one term to the right and then simplify inside the parentheses.

9.6 Ratio and Root Tests

Theorem 9.2.4 states that if a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. That is, the terms of $\{a_n\}$ must get very small. Not only must the terms approach 0, they must approach 0 "fast enough": while $\lim_{n\to\infty} 1/n = 0$, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as the terms of $\{1/n\}$ do not approach 0 "fast enough."

The comparison tests of Section 9.4 determine convergence by comparing terms of a series to terms of another series whose convergence is known. This section introduces the Ratio and Root Tests, which determine convergence by analyzing the terms of a series to see if they approach 0 "fast enough."

Ratio Test

Theorem 9.6.1 Ratio Test

Let $\{a_n\}$ be a sequence where $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Ratio Test is inconclusive.

The principle of the Ratio Test is this: if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then for large n, each term of $\{a_n\}$ is significantly smaller than its previous term which is enough to ensure convergence. A full proof can be found at `http://tutorial.math.lamar.edu/Classes/CalcII/RatioTest.aspx`.

Watch the video:
Using the Ratio Test to Determine if a Series Converges #1 at
`https://youtu.be/iy8mhbZTY7g`

Notes:

Example 9.6.1 Applying the Ratio Test

Use the Ratio Test to determine the convergence of the following series:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} &= \lim_{n\to\infty} \frac{2^{n+1}n!}{2^n(n+1)!}\\ &= \lim_{n\to\infty} \frac{2}{n+1}\\ &= 0.\end{aligned}$$

Since the limit is $0<1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{3^{n+1}/(n+1)^3}{3^n/n^3} &= \lim_{n\to\infty} \frac{3^{n+1}n^3}{3^n(n+1)^3}\\ &= \lim_{n\to\infty} \frac{3n^3}{(n+1)^3}\\ &= 3.\end{aligned}$$

Since the limit is $3>1$, by the Ratio Test $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$:

$$\begin{aligned}\lim_{n\to\infty} \frac{1/\big((n+1)^2+1\big)}{1/(n^2+1)} &= \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\\ &= 1.\end{aligned}$$

Since the limit is 1, the Ratio Test is inconclusive. We can easily show this series converges using the Direct or Limit Comparison Tests, with each comparing to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

The Ratio Test is not effective when the terms of a series *only* contain algebraic functions (e.g., polynomials). It is most effective when the terms contain some factorials or exponentials. The previous example also reinforces our developing intuition: factorials dominate exponentials, which dominate algebraic functions,

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which dominate logarithmic functions. In Part 1 of the example, the factorial in the denominator dominated the exponential in the numerator, causing the series to converge. In Part 2, the exponential in the numerator dominated the algebraic function in the denominator, causing the series to diverge.

While we have used factorials in previous sections, we have not explored them closely and one is likely to not yet have a strong intuitive sense for how they behave. The following example gives more practice with factorials.

Example 9.6.2 Applying the Ratio Test

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

Solution Before we begin, be sure to note the difference between $(2n)!$ and $2n!$. When $n = 4$, the former is $8! = 8 \cdot 7 \cdot \cdots \cdot 2 \cdot 1 = 40{,}320$, whereas the latter is $2(4 \cdot 3 \cdot 2 \cdot 1) = 48$.

Applying the Ratio Test:

$$\lim_{n\to\infty} \frac{(n+1)!(n+1)!/\big(2(n+1)\big)!}{n!n!/(2n)!} = \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{n!n!(2n+2)!}$$

Noting that $(2n+2)! = (2n+2) \cdot (2n+1) \cdot (2n)!$, we have

$$\begin{aligned} &= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \\ &= 1/4. \end{aligned}$$

Since the limit is $1/4 < 1$, by the Ratio Test we conclude $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ converges.

Root Test

The final test we introduce is the Root Test, which works particularly well on series where each term is raised to a power, and does not work well with terms containing factorials.

Note: We won't go into the proof, but the idea (as with the Ratio Test) is that the series is behaving enough like the geometric series $\sum_{n=0}^{\infty} L^n$ that we can determine convergence.

Notes:

Theorem 9.6.2 Root Test
Let $\{a_n\}$ be a sequence where $\lim_{n\to\infty} |a_n|^{1/n} = L$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the Root Test is inconclusive.

Note: We can use L'Hôpital's Rule to show that $n^{1/n}$ approaches 1. If someone insists on using the Root Test with factorials, it can be useful to know that $(n!)^{1/n}$ approaches infinity.

Example 9.6.3 Applying the Root Test
Determine the convergence of the following series using the Root Test:

1. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3n+1}{5n-2}\right)^n$ 2. $\displaystyle\sum_{n=2}^{\infty} \frac{n^4}{(\ln n)^n}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Solution

1. $\displaystyle\lim_{n\to\infty} \left(\left(\frac{3n+1}{5n-2}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{3n+1}{5n-2} = \frac{3}{5}.$

   Since the limit is less than 1, we conclude the series converges. Note: it is difficult to apply the Ratio Test to this series.

2. $\displaystyle\lim_{n\to\infty} \left(\frac{n^4}{(\ln n)^n}\right)^{1/n} = \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n}.$

   As n grows, the numerator approaches 1 (apply L'Hôpital's Rule) and the denominator grows to infinity. Thus

   $$\lim_{n\to\infty} \frac{\left(n^{1/n}\right)^4}{\ln n} = 0.$$

   Since the limit is less than 1, we conclude the series converges.

3. $\displaystyle\lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{\left(n^{1/n}\right)^2} = 2.$

   Since this is greater than 1, we conclude the series diverges.

Notes:

We end here our study of tests to determine convergence. The next section of this text provides strategies for testing series, while the back of the book contains a table summarizing the tests that one may find useful.

While series are worthy of study in and of themselves, our ultimate goal within calculus is the study of Power Series, which we will consider in Section 9.8. We will use power series to create functions where the output is the result of an infinite summation.

Notes:

Exercises 9.6

Terms and Concepts

1. The Ratio Test is not effective when the terms of a sequence only contain ______ functions.
2. The Ratio Test is most effective when the terms of a sequence contains ______ and/or ______ functions.
3. What three convergence tests do not work well with terms containing factorials?
4. The Root Test works particularly well on series where each term is ______ to a ______.

Problems

In Exercises 5–16, determine the convergence of the given series using the Ratio Test. If the Ratio Test is inconclusive, state so and determine convergence with another test.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n}{n!}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{n!10^n}{(2n)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + n^4}{7^n + n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^3 + 7}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{10 \cdot 5^n}{7^n - 3}$

12. $\displaystyle\sum_{n=1}^{\infty} n \cdot \left(\frac{3}{5}\right)^n$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots 2n}{3 \cdot 6 \cdot 9 \cdot 12 \cdots 3n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5 \cdot 10 \cdot 15 \cdots (5n)}$

15. $\displaystyle\sum_{n=1}^{\infty} e^{-n} n!$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$

In Exercises 17–26, determine the convergence of the given series using the Root Test. If the Root Test is inconclusive, state so and determine convergence with another test.

17. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+5}{3n+11}\right)^n$

18. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{.9n^2 - n - 3}{n^2 + n + 3}\right)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{3^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+7}}{7^n}$

23. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 - n}{n^2 + n}\right)^n$

24. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(\ln n)^n}$

27. We know that the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges. Suppose we remove some terms by considering the series $\sum_{n=1}^{\infty} \frac{1}{F_n}$ where F_n is the n^{th} Fibonacci number (so $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$, ..., and in general $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$). Determine if this series converges or diverges, using the fact that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$ is known as the Golden Ratio.

9.7 Strategy for Testing Series

We have now covered all of the tests for determining the convergence or divergence of a series, which we summarize here. Because more than one test may apply to a given series, you should always go completely through the guidelines and identify all possible tests that you can use. Once you've done this, you can identify the test that will be the easiest for you to use.

1. With a quick glance does it look like the series terms don't converge to zero in the limit, i.e. does $\lim\limits_{n\to\infty} a_n \neq 0$? If so, use the Test for Divergence. Note that you should only use the Test for Divergence if a quick glance suggests that the series terms may not converge to zero in the limit.

2. Is the series a p-series $\left(\sum n^{-p}\right)$ or a geometric series $\left(\sum ar^n\right)$? If so, use the fact that p-series will converge only if $p > 1$ and a geometric series will only converge if $|r| < 1$. Remember as well that often some algebraic manipulation is required to get a geometric series into the correct form.

3. Is the series similar to a p-series or a geometric series? If so, try the Comparison Test.

4. Is the series a rational expression involving only polynomials or polynomials under radicals? If so, try the Comparison test or the Limit Comparison Test. Remember however, that in order to use the Comparison Test and the Limit Comparison Test the series terms all need to be positive.

5. Is the series of the form $\sum(-1)^n a_n$? If so, then the Alternating Series Test may work.

6. Does the series contain factorials or constants raised to powers involving n? If so, then the Ratio Test may work. Note that if the series term contains a factorial then the only test that we have that will work is the Ratio Test. (If you find that $L > 1$, then the divergence test also would have worked.)

7. Can the series terms be written in the form $a_n = (b_n)^n$? If so, then the Root Test may work.

8. If $a_n = f(n)$ for some positive, decreasing function and $\displaystyle\int_a^\infty f(x)\, dx$ is easy to evaluate then the Integral Test may work.

Again, remember that these are only a set of guidelines and not a set of hard and fast rules to use when trying to determine the best test to use on a series. If more that one test can be used, try to use the test that will be the easiest for

Notes:

you to use. These guidelines are also summarized in a table in the back of the book.

We now consider several examples.

Example 9.7.1 **Testing Series**

Determine whether the given series converges absolutely, converges conditionally, or diverges.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+3}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3n}{4n^2-2n+1}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{e^n}{(n+3)!}$

Solution

1. We see that this series is alternating so we use the alternating series test. The underlying sequence is $\{a_n\} = \{\frac{n}{n^2+3}\}$ which is positive and decreasing since $a'(n) = \dfrac{3-n^2}{(n^2+3)^2} < 0$ for $n \geq 2$. We also see $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+3} = 0$ so by the Alternating Series Test $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+3}$ converges. We now determine if it converges absolutely. Consider the series $\displaystyle\sum_{n=2}^{\infty} \left|\frac{(-1)^n n}{n^2+3}\right| = \sum_{n=2}^{\infty} \frac{n}{n^2+3}$. Using the Limit Comparison Test with the divergent p-series $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n}$, $\displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2+3}$ diverges. Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+3}$ converges conditionally.
2. $\displaystyle\lim_{n\to\infty} \frac{n^2-3n}{4n^2-2n+1} = \frac{1}{4}$ so by the Test for Divergence $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3n}{4n^2-2n+1}$ diverges.

Notes:

3. We see the factorial and use the Ratio Test. All terms of the series are positive so we consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+4)!}}{\frac{e^n}{(n+3)!}} \\
&= \lim_{n\to\infty} \frac{e^{n+1}(n+3)!}{e^n(n+4)!} \\
&= \lim_{n\to\infty} \frac{e\cdot e^n(n+3)!}{e^n(n+4)(n+3)!} \\
&= \lim_{n\to\infty} \frac{e}{n+4} = 0 < 1
\end{aligned}$$

So by the Ratio Test, $\displaystyle\sum_{n=2}^{\infty} \frac{e^n}{(n+3)!}$ converges. Because all of the series terms are positive it converges absolutely.

Notes:

Exercises 9.7

Problems

In Exercises 1–38, determine whether the given series converges absolutely, converges conditionally, or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n(n+2)(n+4)}}$

2. $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^{n-1}$

3. $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n5^{n-1}}$

4. $\sum_{n=1}^{\infty} n^{-2} e^{\frac{1}{n}}$

5. $\sum_{n=1}^{\infty} \frac{n!}{\ln(n+2)}$

6. $\sum_{n=1}^{\infty} (n^2+4)(-2)^{1-n}$

7. $\sum_{n=1}^{\infty} \frac{2}{n+4^n}$

8. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$

10. $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{4\pi n}{3}\right)}{n^{4\pi/3}}$

11. $\sum_{n=1}^{\infty} \frac{3^n n!}{(n+2)!}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

13. $\sum_{n=1}^{\infty} \frac{1-\cos n}{n^3}$

14. $\sum_{n=1}^{\infty} \frac{4+3n-5n^3}{2+n^3}$

15. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{n^4+1}$

16. $\sum_{n=1}^{\infty} \frac{(3n)^n}{n^{3n}}$

17. $\sum_{n=1}^{\infty} \frac{e^{2n}}{(2n-1)!}$

18. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

19. $\sum_{n=2}^{\infty} (-1)^{n-1} \frac{\sqrt{\ln n}}{n}$

20. $\sum_{n=1}^{\infty} \frac{n^2}{(-2)^n}$

21. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+5)^3}$

22. $\sum_{n=1}^{\infty} \frac{n!}{(-10)^n}$

23. $\sum_{n=1}^{\infty} \frac{\ln n}{n!}$

24. $\sum_{n=1}^{\infty} \frac{1}{(-3)^n+n}$

25. $\sum_{n=1}^{\infty} \frac{(-1)^n(n-2)}{10n+5}$

26. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

27. $\sum_{n=1}^{\infty} \frac{\cos(1/n)}{\sqrt{n}}$

28. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2+4n-2)}{n^3+4n^2-3n+7}$

29. $\sum_{n=1}^{\infty} \frac{n^4(-4)^n}{n!}$

30. $\sum_{n=1}^{\infty} \frac{n^2}{(-3)^n+n}$

31. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+4n+1}}$

32. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^n}$

33. $\sum_{n=1}^{\infty} \frac{n!n!n!}{(3n)!}$

34. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

35. $\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^n$

36. $\sum_{n=2}^{\infty} \frac{n^3}{(\ln n)^n}$

37. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$

38. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

9.8 Power Series

So far, our study of series has examined the question of "Is the sum of these infinite terms finite?," i.e., "Does the series converge?" We now approach series from a different perspective: as a function. Given a value of x, we evaluate $f(x)$ by finding the sum of a particular series that depends on x (assuming the series converges). We start this new approach to series with a definition.

Definition 9.8.1 Power Series

Let $\{a_n\}$ be a sequence, let x be a variable, and let c be a real number.

1. The **power series in** x is the series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

2. The **power series in** x **centered at** c is the series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Example 9.8.1 Examples of power series

Write out the first five terms of the following power series:

1. $\displaystyle\sum_{n=0}^{\infty} x^n$ 2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x+1)^n}{n}$ 3. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$.

Solution

1. One of the conventions we adopt is that $x^0 = 1$ regardless of the value of x. Therefore
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
This is a geometric series in x.

2. This series is centered at $c = -1$. Note how this series starts with $n = 1$. We could rewrite this series starting at $n = 0$ with the understanding that

Notes:

$a_0 = 0$, and hence the first term is 0.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x+1)^n}{n}$$
$$= (x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \frac{(x+1)^4}{4} + \frac{(x+1)^5}{5}\cdots$$

3. This series is centered at $c = \pi$. Recall that $0! = 1$.

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}$$
$$= -1 + \frac{(x-\pi)^2}{2} - \frac{(x-\pi)^4}{24} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!}\cdots$$

We introduced power series as a type of function, where a value of x is given and the sum of a series is returned. Of course, not every series converges. For instance, in part 1 of Example 9.8.1, we recognized the series $\displaystyle\sum_{n=0}^{\infty} x^n$ as a geometric series in x. Theorem 9.2.1 states that this series converges only when $|x| < 1$.

This raises the question: "For what values of x will a given power series converge?," which leads us to a theorem and definition.

Theorem 9.8.1 Convergence of Power Series

Let a power series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ be given. Then one of the following is true:

1. The series converges only at $x = c$.
2. There is an $R > 0$ such that the series converges for all x in $(c-R, c+R)$ and diverges for all $x < c-R$ and $x > c+R$.
3. The series converges for all x.

The value of R is important when understanding a power series, hence it is given a name in the following definition. Also, note that part 2 of Theorem 9.8.1 makes a statement about the interval $(c-R, c+R)$, but the not the endpoints of that interval. A series may or may not converge at these endpoints.

Notes:

Definition 9.8.2 Radius and Interval of Convergence

1. The number R given in Theorem 9.8.1 is the **radius of convergence** of a given series. When a series converges for only $x = c$, we say the radius of convergence is 0, i.e., $R = 0$. When a series converges for all x, we say the series has an infinite radius of convergence, i.e., $R = \infty$.

2. The **interval of convergence** is the set of all values of x for which the series converges.

To find the values of x for which a given series converges, we will use the convergence tests we studied previously (especially the Ratio Test). However, the tests all required that the terms of a series be positive. The following theorem gives us a work-around to this problem.

Theorem 9.8.2 The Radius of Convergence of a Series and Absolute Convergence

The series $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ and $\displaystyle\sum_{n=0}^{\infty} |a_n(x-c)^n|$ have the same radius of convergence R.

Theorem 9.8.2 allows us to find the radius of convergence R of a series by applying the Ratio Test (or any applicable test) to the absolute value of the terms of the series. We practice this in the following example.

Watch the video:
Power Series — Finding the Interval of Convergence at
`https://youtu.be/01LzAU__J-0`

Example 9.8.2 Determining the radius and interval of convergence.

Find the radius and interval of convergence for each of the following series:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$
3. $\displaystyle\sum_{n=0}^{\infty} 2^n(x-3)^n$
4. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

Notes:

Solution

1. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty}\left|\frac{x^n}{n!}\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)!\right|}{\left|x^n/n!\right|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right|\\ &= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|\\ &= 0 \text{ for all } x.\end{aligned}$$

The Ratio Test shows us that regardless of the choice of x, the series converges. Therefore the radius of convergence is $R=\infty$, and the interval of convergence is $(-\infty,\infty)$.

2. We apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{x^n}{n}\right| = \sum_{n=1}^{\infty}\left|\frac{x^n}{n}\right|$:

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|x^{n+1}/(n+1)\right|}{\left|x^n/n\right|} &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n}{n+1}\right|\\ &= \lim_{n\to\infty}|x|\,\frac{n}{n+1}\\ &= |x|\,.\end{aligned}$$

The Ratio Test states a series converges if the limit of $|a_{n+1}/a_n| = L < 1$. We found the limit above to be $|x|$; therefore, the power series converges when $|x|<1$, or when x is in $(-1,1)$. Thus the radius of convergence is $R=1$.

To determine the interval of convergence, we need to check the endpoints of $(-1,1)$. When $x=-1$, we have the opposite of the Harmonic Series:

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n} &= \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}\\ &= \sum_{n=1}^{\infty}\frac{-1}{n}\\ &= -\infty.\end{aligned}$$

The series diverges when $x=-1$.

When $x=1$, we have the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n}$, which is the Alternating Harmonic Series, which converges. Therefore the interval of convergence is $(-1,1]$.

Notes:

3. We apply the Ratio Test to the series $\displaystyle\sum_{n=0}^{\infty} \left|2^n (x-3)^n\right|$:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|2^{n+1}(x-3)^{n+1}\right|}{\left|2^n(x-3)^n\right|} &= \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n}\cdot\frac{(x-3)^{n+1}}{(x-3)^n}\right| \\ &= \lim_{n\to\infty} \left|2(x-3)\right|.\end{aligned}$$

According to the Ratio Test, the series converges when $|2(x-3)| < 1 \implies |x-3| < 1/2$. The series is centered at 3, and x must be within $1/2$ of 3 in order for the series to converge. Therefore the radius of convergence is $R = 1/2$, and we know that the series converges absolutely for all x in $(3-1/2, 3+1/2) = (2.5, 3.5)$.

We check for convergence at the endpoints to find the interval of convergence. When $x = 2.5$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} 2^n(2.5-3)^n &= \sum_{n=0}^{\infty} 2^n(-1/2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n,\end{aligned}$$

which diverges. A similar process shows that the series also diverges at $x = 3.5$. Therefore the interval of convergence is $(2.5, 3.5)$.

4. We apply the Ratio Test to $\displaystyle\sum_{n=0}^{\infty} \left|n!x^n\right|$:

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|(n+1)!x^{n+1}\right|}{\left|n!x^n\right|} &= \lim_{n\to\infty} \left|(n+1)x\right| \\ &= \infty \text{ for all } x\text{, except } x = 0.\end{aligned}$$

The Ratio Test shows that the series diverges for all x except $x = 0$. Therefore the radius of convergence is $R = 0$.

Power Series as Functions

We can use a power series to define a function:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Notes:

where the domain of f is a subset of the interval of convergence of the power series. One can apply calculus techniques to such functions; in particular, we can find derivatives and antiderivatives.

Theorem 9.8.3 Derivatives and Indefinite Integrals of Power Series Functions

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ be a function defined by a power series, with radius of convergence R.

1. $f(x)$ is continuous and differentiable on $(c-R, c+R)$.
2. $f'(x) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1}$, with radius of convergence R.
3. $\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$, with radius of convergence R.

A few notes about Theorem 9.8.3:

1. The theorem states that differentiation and integration do not change the radius of convergence. It does not state anything about the *interval* of convergence. They are not always the same.
2. Notice how the summation for $f'(x)$ starts with $n=1$. This is because the constant term a_0 of $f(x)$ goes to 0.
3. Differentiation and integration are simply calculated term-by-term using previous rules of integration and differentiation.

Example 9.8.3 Derivatives and indefinite integrals of power series

Let $f(x) = \sum_{n=0}^{\infty} x^n$. Find the following along with their respective intervals of convergence.

1. $f'(x)$ and 2. $F(x) = \int f(x)\,\mathrm{d}x$

Solution We find the derivative and indefinite integral of $f(x)$, following Theorem 9.8.3.

Notes:

1. $$f(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$f'(x) = 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$$

In Example 9.8.1, we recognized that $\sum_{n=0}^{\infty} x^n$ is a geometric series in x. We know that such a geometric series converges when $|x| < 1$; that is, the interval of convergence is $(-1, 1)$.

To determine the interval of convergence of $f'(x)$, we consider the endpoints of $(-1, 1)$. When $x = -1$ we have

$$f'(-1) = \sum_{n=1}^{\infty} n(-1)^{n-1}$$

which diverges by the Test for Divergence and when $x = 1$ we have

$$f'(1) = \sum_{n=1}^{\infty} n$$

which also diverges by the Test for Divergence. Therefore, the interval of convergence of $f'(x)$ is $(-1, 1)$.

2. $$f(x) = \qquad 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$F(x) = \int f(x)\,\mathrm{d}x = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty} \frac{x^n}{n}$$

To find the interval of convergence of $F(x)$, we again consider the endpoints of $(-1, 1)$. When $x = -1$ we have

$$F(-1) = C + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

The value of C is irrelevant; notice that the rest of the series is an Alternating Series that whose terms converge to 0. By the Alternating Series Test, this series converges. (In fact, we can recognize that the terms of the

Notes:

series after C are the opposite of the Alternating Harmonic Series. We can thus say that $F(-1) = C - \ln 2$.)

$$F(1) = C + \sum_{n=1}^{\infty} \frac{1}{n}$$

Notice that this summation is $C +$ the Harmonic Series, which diverges. Since F converges for $x = -1$ and diverges for $x = 1$, the interval of convergence of $F(x)$ is $[-1, 1)$.

The previous example showed how to take the derivative and indefinite integral of a power series without motivation for why we care about such operations. We may care for the sheer mathematical enjoyment "that we can", which is motivation enough for many. However, we would be remiss to not recognize that we can learn a great deal from taking derivatives and indefinite integrals.

Recall that $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$ in Example 9.8.3 is a geometric series. According to Theorem 9.2.1, this series converges to $1/(1-x)$ when $|x| < 1$. Thus we can say

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \text{on} \quad (-1, 1). \tag{9.8.1}$$

Integrating the power series, (as done in Example 9.8.3,) we find

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \tag{9.8.2}$$

while integrating the function $f(x) = 1/(1-x)$ gives

$$F(x) = -\ln|1-x| + C_2. \tag{9.8.3}$$

Equating Equations (9.8.2) and (9.8.3), we have

$$F(x) = C_1 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C_2.$$

Letting $x = 0$, we have $F(0) = C_1 = C_2$. This implies that we can drop the constants and conclude

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

Notes:

We established in Example 9.8.3 that the series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ converges at $x = -1$; substituting $x = -1$ on both sides of the above equality gives

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = -\ln 2.$$

On the left we have the opposite of the Alternating Harmonic Series; on the right, we have $-\ln 2$. We conclude that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

In Example 9.5.1 of Section 9.5 we said the Alternating Harmonic Series converges to $\ln 2$, but did not show why this was the case. The work above shows how we conclude that the Alternating Harmonic Series Converges to $\ln 2$.

We use this type of analysis in the next example.

Example 9.8.4 Analyzing power series functions

Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Find $f'(x)$ and $\int f(x)\,dx$, and use these to analyze the behavior of $f(x)$.

Solution We start by making two notes: first, in Example 9.8.2, we found the interval of convergence of this power series is $(-\infty, \infty)$. Second, we will find it useful later to have a few terms of the series written out:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \qquad (9.8.4)$$

We now find the derivative:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = 1 + x + \frac{x^2}{2!} + \cdots . \end{aligned}$$

Since the series starts at $n = 1$ and each term refers to $(n-1)$, we can re-index the series starting with $n = 0$:

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= f(x). \end{aligned}$$

Notes:

We found the derivative of $f(x)$ is $f(x)$. The only functions for which this is true are of the form $y = ce^x$ for some constant c. As $f(0) = 1$ (see Equation (9.8.4)), c must be 1. Therefore we conclude that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

for all x.

We can also find $\displaystyle\int f(x)\,\mathrm{d}x$:

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)} \\ &= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}\end{aligned}$$

We write out a few terms of this last series:

$$C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The integral of $f(x)$ differs from $f(x)$ only by a constant, again indicating that $f(x) = e^x$.

Example 9.8.4 and the work following Example 9.8.3 established relationships between a power series function and "regular" functions that we have dealt with in the past. In general, given a power series function, it is difficult (if not impossible) to express the function in terms of elementary functions. We chose examples where things worked out nicely.

Representations of Functions with Power Series

It can be difficult to recognize an elementary function by its power series expansion. It is far easier to start with a known function, expressed in terms of elementary functions, and represent it as a power series function. One may wonder why we would bother doing so, as the latter function probably seems more complicated.

Let's start off with a series we already know how to do, although when we first ran across this series we didn't think of it as a power series nor did we acknowledge that it represented a function. Recall that the geometric series is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{provided } |r| < 1.$$

Notes:

We also know that if $|r| \geq 1$ the series diverges. Now, if we take $a = 1$ and $r = x$ this becomes,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \text{provided } |x| < 1 \tag{9.8.5}$$

Turning this around we can see that we can represent the function

$$f(x) = \frac{1}{1-x} \tag{9.8.6}$$

with the power series

$$\sum_{n=0}^{\infty} x^n \qquad \text{provided } |x| < 1. \tag{9.8.7}$$

This provision is important. We can clearly plug any number other than $x = 1$ into the function, however, we will only get a convergent power series if $|x| < 1$. This means the equality in Equation (9.8.5) will only hold if $|x| < 1$. For any other value of x the equality won't hold. Note as well that we can also use this to acknowledge that the radius of convergence of this power series is $R = 1$ and the interval of convergence is $|x| < 1$.

This idea of convergence is important here. We will be representing many functions as power series and it will be important to recognize that the representations will often only be valid for a range of x's and that there may be values of x that we can plug into the function that we can't plug into the power series representation.

In this section we are going to concentrate on representing functions with power series where the function can be related back to a geometric series. In this way we will hopefully become familiar with some of the kinds of manipulations that we will sometimes need when working with power series. We will see in Section 9.10 that this strategy is useful for integrating functions that don't have elementary antiderivatives.

Example 9.8.5 Finding a Power Series

Find a power series representation for $g(x) = \dfrac{1}{1+x^3}$ and determine its interval of convergence.

Solution We want to relate this function back to Equation (9.8.6). This is actually easier than it might look. Recall that the x in Equation (9.8.6) is simply a variable and can represent anything. So, a quick rewrite of $g(x)$ gives,

$$g(x) = \frac{1}{1-(-x^3)}$$

Notes:

and so the $-x^3$ holds the same place as the x in Equation (9.8.6). Therefore, all we need to do is replace the x in Equation (9.8.7) and we've got a power series representation for $g(x)$.

$$g(x) = \sum_{n=0}^{\infty} \left(-x^3\right)^n \qquad \text{provided } \left|-x^3\right| < 1$$

Notice that we replaced both the x in the power series and in the interval of convergence. All we need to do now is a little simplification.

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x| < 1$$

So, in this case the interval of convergence is the same as the original power series. This usually won't happen. More often than not the new interval of convergence will be different from the original interval of convergence.

Example 9.8.6 Finding a Power Series

Find a power series representation for $h(x) = \dfrac{2x^2}{1+x^3}$ and determine its interval of convergence.

Solution This function is similar to the previous function, however the numerator is different. Since Equation (9.8.6) doesn't have an x in the numerator it appears that we can't relate this function back to that. However, now that we've worked the first example this one is actually very simple since we can use the result of the answer from that example. To see how to do this let's first rewrite the function a little.

$$h(x) = 2x^2 \frac{1}{1+x^3}.$$

Now, from the first example we've already got a power series for the second term so let's use that to write the function as,

$$h(x) = 2x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad \text{provided } |x| < 1$$

Notice that the presence of x's outside of the series will NOT affect its convergence and so the interval of convergence remains the same. The last step is to bring the coefficient into the series and we'll be done. When we do this, make sure to combine the x's as well. We typically only want a single x in a power

Notes:

series.

$$h(x) = \sum_{n=0}^{\infty} 2\,(-1)^n\, x^{3n+2} \qquad \text{provided } |x| < 1.$$

As we saw in the previous example we can often use previous results to help us out. This is an important idea to remember as it can often greatly simplify our work.

Example 9.8.7 Finding a Power Series

Find a power series representation for $f(x) = \dfrac{x}{5-x}$ and determine its interval of convergence.

Solution So again, we have an x in the numerator. As with the last example factor x out and we have $f(x) = x\dfrac{1}{5-x}$. If we had a power series representation for $g(x) = \dfrac{1}{5-x}$ we could get a power series representation for $f(x)$. We need the number in the denominator to be a one so we rewrite the denominator.

$$g(x) = \frac{1}{5}\frac{1}{1-\frac{x}{5}}$$

Now all we need to do to get a power series representation is to replace the x in Equation (9.8.7) with $\dfrac{x}{5}$. Doing this gives

$$g(x) = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n \qquad \text{provided } \left|\frac{x}{5}\right| < 1.$$

Now simplify the series.

$$\begin{aligned} g(x) &= \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^n}{5^n} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \end{aligned}$$

The interval of convergence for this series is

$$\left|\frac{x}{5}\right| < 1 \qquad \Rightarrow \qquad \frac{1}{5}|x| < 1 \qquad \Rightarrow \qquad |x| < 5$$

Notes:

We now have a power series representation for $g(x)$ but we need to find a power series representation for the original function. All we need to do for this is to multiply the power series representative for $g(x)$ by x and we'll have it.

$$\begin{aligned} f(x) &= x\frac{1}{5-x} \\ &= x\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{x^{n+1}}{5^{n+1}} \end{aligned}$$

The interval of convergence doesn't change and so it will be $|x| < 5$.

Example 9.8.8 Re-indexing a Power Series

Find a power series representation for $f(x) = \dfrac{1+x}{1-x}$.

Solution We can start by writing this as

$$f(x) = (1+x)\sum_{n=0}^{\infty}x^n.$$

The problem with this representation is that the $1+x$ makes this not a power series — we only want a single occurrence of x. The proceed, we'll split this into two sums

$$f(x) = \sum_{n=0}^{\infty}x^n + x\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}x^n + \sum_{n=0}^{\infty}x^{n+1}.$$

Because the second sum uses $n+1$, we can re-index it to get a sum involving x^n, which is our power series

$$f(x) = \sum_{n=0}^{\infty}x^n + x\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}x^n + \sum_{n=1}^{\infty}x^n = 1 + 2\sum_{n=1}^{\infty}x^n.$$

We now consider several examples where differentiation and integration of power series from Theorem 9.8.3 are used to write the power series for a function.

Notes:

Example 9.8.9 Differentiating a Power Series

Find a power series representation for $g(x) = \dfrac{1}{(1-x)^2}$ and determine its radius of convergence.

Solution We know that

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right).$$

Since we have a power series representation for $\dfrac{1}{1-x}$, we can differentiate that power series to get a power series representation for $g(x)$.

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^2} \\ &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ &= \sum_{n=1}^{\infty} nx^{n-1} \end{aligned}$$

Since the original power series had a radius of convergence of $R = 1$ the derivative, and hence $g(x)$, will also have a radius of convergence of $R = 1$.

Example 9.8.10 Integrating a Power Series

Find a power series representation for $h(x) = \ln(5-x)$ and determine its radius of convergence.

Solution In this case we need the fact that

$$\int \frac{1}{5-x}\,dx = -\ln(5-x).$$

Recall that we found a power series representation for $\dfrac{1}{5-x}$ in Example 9.8.7.

Notes:

We now have

$$\begin{aligned}
\ln(5-x) &= -\int \frac{1}{5-x}\,dx \\
&= -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}\,dx \qquad \text{where } |x| < 5 \\
&= C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}} \qquad \text{where } |x| < 5
\end{aligned}$$

We can find the constant of integration, C, by substituting in a value of x. A good choice is $x = 0$ as the series is usually easy to evaluate there.

$$\begin{aligned}
\ln(5-0) &= C - \sum_{n=0}^{\infty} \frac{0^{n+1}}{(n+1)5^{n+1}} \\
\ln(5-0) &= C
\end{aligned}$$

So, the final answer is,

$$\ln(5-x) = \ln(5) - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}},$$

and the radius of convergence is 5. Notice that $x = -5$ allows for convergence so the interval of convergence is $[-5, 5)$.

Notes:

Exercises 9.8

Terms and Concepts

1. We adopt the convention that x^0 =______, regardless of the value of x.
2. What is the difference between the radius of convergence and the interval of convergence?
3. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=1}^{\infty} n \cdot a_n x^{n-1}$?
4. If the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 5, what is the radius of convergence of $\sum_{n=0}^{\infty} (-1)^n a_n x^n$?

Problems

In Exercises 5–8, write out the sum of the first 5 terms of the given power series.

5. $\sum_{n=0}^{\infty} 2^n x^n$
6. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$
7. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$
8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

In Exercises 9–28, a power series is given.

(a) Find the radius of convergence.

(b) Find the interval of convergence.

9. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} x^n$
10. $\sum_{n=0}^{\infty} n x^n$
11. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n}$
12. $\sum_{n=0}^{\infty} \frac{(x+4)^n}{n!}$
13. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$
14. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{10^n}$
15. $\sum_{n=0}^{\infty} 5^n (x-1)^n$
16. $\sum_{n=0}^{\infty} (-2)^n x^n$
17. $\sum_{n=0}^{\infty} \sqrt{n} x^n$
18. $\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$
19. $\sum_{n=0}^{\infty} \frac{3^n}{n!} (x-5)^n$
20. $\sum_{n=0}^{\infty} (-1)^n n! (x-10)^n$
21. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$
22. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^3}$
23. $\sum_{n=0}^{\infty} n! \left(\frac{x}{10}\right)^n$
24. $\sum_{n=0}^{\infty} n^2 \left(\frac{x+4}{4}\right)^n$
25. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$
26. $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$
27. $\sum_{n=2}^{\infty} \frac{x^n}{(\ln n)^n}$
28. $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

In Exercises 29–32, write the following functions as a series and give the radius of convergence.

29. $f(x) = \frac{x}{1-8x}$
30. $f(x) = \frac{6}{1+7x^4}$
31. $f(x) = \frac{x^3}{3-x^2}$
32. $f(x) = \frac{3x^2}{5-2\sqrt[3]{x}}$

In Exercises 33–44, a function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is given.

(a) Give a power series for $f'(x)$ and its interval of convergence.

(b) Give a power series for $\int f(x)\, dx$ and its interval of convergence.

33. $\sum_{n=0}^{\infty} n x^n$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

35. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

36. $\displaystyle\sum_{n=0}^{\infty} (-3x)^n$

37. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

38. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

39. $\displaystyle\sum_{n=0}^{\infty} nx^n$

40. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

41. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

42. $\displaystyle\sum_{n=0}^{\infty} (-3x)^n$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

45. (a) Use differentiation to find a power series representation for $f(x) = \dfrac{1}{(1+x)^2}$. What is the radius of convergence?

(b) Use part (a) to find a power series for $f(x) = \dfrac{1}{(1+x)^3}$.

(c) Use part (b) to find a power series for $f(x) = \dfrac{x^2}{(1+x)^3}$.

46. Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges for $x = -3$ and diverges when $x = 7$. What can you say about the convergence or divergence of the following series?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n$

(b) $\displaystyle\sum_{n=0}^{\infty} c_n 9^n$

(c) $\displaystyle\sum_{n=0}^{\infty} c_n (-2)^n$

(d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 8^n$

In Exercises 47–53, find a power series representation for the function and determine the radius of convergence.

47. $f(x) = \ln(3 - x)$

48. $f(x) = \dfrac{x}{(1+9x)^2}$

49. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

50. $f(x) = \tan^{-1} x$

51. $f(x) = x^2 \tan^{-1}(x^3)$

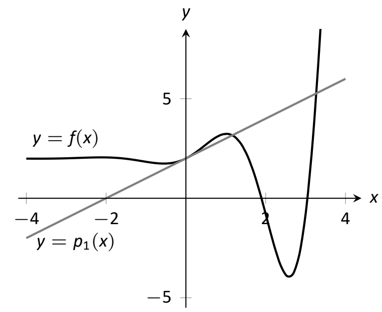
52. $f(x) = \dfrac{1+x}{(1-x)^2}$

53. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

9.9 Taylor Polynomials

Consider a function $y = f(x)$ and a point $\big(c, f(c)\big)$. The derivative, $f'(c)$, gives the instantaneous rate of change of f at $x = c$. Of all lines that pass through the point $\big(c, f(c)\big)$, the line that best approximates f at this point is the tangent line; that is, the line whose slope (rate of change) is $f'(c)$.

In Figure 9.9.1, we see a function $y = f(x)$ graphed. The table below the graph shows that $f(0) = 2$ and $f'(0) = 1$; therefore, the tangent line to f at $x = 0$ is $p_1(x) = 1(x-0)+2 = x+2$. The tangent line is also given in the figure. Note that "near" $x = 0$, $p_1(x) \approx f(x)$; that is, the tangent line approximates f well.



$f(0) = 2$	$f'''(0) = -1$
$f'(0) = 1$	$f^{(4)}(0) = -12$
$f''(0) = 2$	$f^{(5)}(0) = -19$

Figure 9.9.1: Plotting $y = f(x)$ and a table of derivatives of f evaluated at 0.

One shortcoming of this approximation is that the tangent line only matches the slope of f; it does not, for instance, match the concavity of f. We can find a polynomial, $p_2(x)$, that does match the concavity without much difficulty, though. The table in Figure 9.9.1 gives the following information:

$$f(0) = 2 \qquad f'(0) = 1 \qquad f''(0) = 2.$$

Therefore, we want our polynomial $p_2(x)$ to have these same properties. That is, we need

$$p_2(0) = 2 \qquad p_2'(0) = 1 \qquad p_2''(0) = 2.$$

This is simply an initial-value problem. We can solve this using the techniques first described in Section 5.1. To keep $p_2(x)$ as simple as possible, we'll assume that not only $p_2''(0) = 2$, but that $p_2''(x) = 2$. That is, the second derivative of p_2 is constant.

If $p_2''(x) = 2$, then $p_2'(x) = 2x + C$ for some constant C. Since we have determined that $p_2'(0) = 1$, we find that $C = 1$ and so $p_2'(x) = 2x + 1$. Finally, we can compute $p_2(x) = x^2 + x + C$. Using our initial values, we know $p_2(0) = 2$ so $C = 2$. We conclude that $p_2(x) = x^2 + x + 2$. This function is plotted with f in Figure 9.9.2.

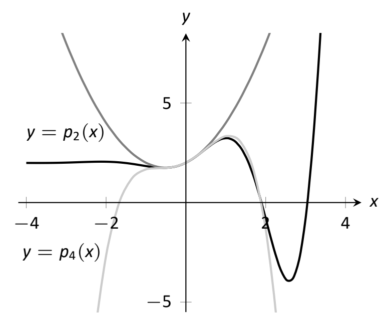


Figure 9.9.2: Plotting f, p_2, and p_4.

We can repeat this approximation process by creating polynomials of higher degree that match more of the derivatives of f at $x = 0$. In general, a polynomial of degree n can be created to match the first n derivatives of f. Figure 9.9.2 also shows $p_4(x) = -x^4/2 - x^3/6 + x^2 + x + 2$, whose first four derivatives at 0 match those of f. (Using the table in Figure 9.9.1, start with $p_4^{(4)}(x) = -12$ and solve the related initial-value problem.)

As we use more and more derivatives, our polynomial approximation to f gets better and better. In this example, the interval on which the approximation is "good" gets bigger and bigger. Figure 9.9.3 shows $p_{13}(x)$; we can visually affirm

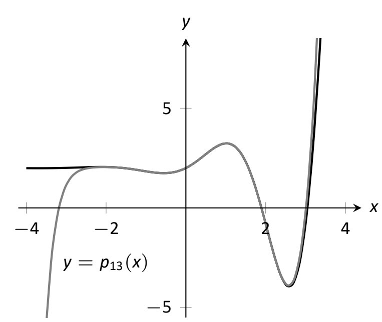


Figure 9.9.3: Plotting f and p_{13}.

Notes:

that this polynomial approximates f very well on $[-2, 3]$. The polynomial $p_{13}(x)$ is fairly complicated:

$$\frac{16901x^{13}}{6227020800}+\frac{13x^{12}}{1209600}-\frac{1321x^{11}}{39916800}-\frac{779x^{10}}{1814400}-\frac{359x^9}{362880}+\frac{x^8}{240}+\frac{139x^7}{5040}+\frac{11x^6}{360}-\frac{19x^5}{120}-\frac{x^4}{2}-\frac{x^3}{6}+x^2+x+2.$$

The polynomials we have created are examples of *Taylor polynomials*, named after the British mathematician Brook Taylor who made important discoveries about such functions. While we created the above Taylor polynomials by solving initial-value problems, it can be shown that Taylor polynomials follow a general pattern that makes their formation much more direct. This is described in the following definition.

Definition 9.9.1 Taylor Polynomial, Maclaurin Polynomial

Let f be a function whose first n derivatives exist at $x = c$.

1. The **Taylor polynomial of degree n of f at $x = c$** is

$$\begin{aligned} p_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k. \end{aligned}$$

2. A special case of the Taylor polynomial is the Maclaurin polynomial, where $c = 0$. That is, the **Maclaurin polynomial of degree n of f** is

$$\begin{aligned} p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k. \end{aligned}$$

Note: The summations in this definition use the convention that $x^0 = 1$ even when $x = 0$ and that $f^{(0)} = f$. They also use the definition that $0! = 1$.

Generally, we order the terms of a polynomial to have decreasing degrees, and that is how we began this section. This definition, and the rest of this chapter, reverses this order to reflect the greater importance of the lower degree terms in the polynomials that we will be finding.

Watch the video:
Taylor Polynomial to Approximate a Function, Ex 3
at
`https://youtu.be/UINFWGOErSA`

Notes:

We will practice creating Taylor and Maclaurin polynomials in the following examples.

Example 9.9.1 Finding and using Maclaurin polynomials

1. Find the n^{th} Maclaurin polynomial for $f(x) = e^x$.
2. Use $p_5(x)$ to approximate the value of e.

SOLUTION

1. We start with creating a table of the derivatives of e^x evaluated at $x = 0$. In this particular case, this is relatively simple, as shown in Figure 9.9.4. By the definition of the Maclaurin polynomial, we have

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{n} \frac{1}{k!}x^k.$$

2. Using our answer from part 1, we have

$$p_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5.$$

To approximate the value of e, note that $e = e^1 = f(1) \approx p_5(1)$. It is very straightforward to evaluate $p_5(1)$:

$$p_5(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60} \approx 2.71667.$$

This is an error of about 0.0016, or 0.06% of the true value.

A plot of $f(x) = e^x$ and $p_5(x)$ is given in Figure 9.9.5.

$$\begin{aligned}
f(x) &= e^x & \Rightarrow && f(0) &= 1\\
f'(x) &= e^x & \Rightarrow && f'(0) &= 1\\
f''(x) &= e^x & \Rightarrow && f''(0) &= 1\\
&\vdots &&& &\vdots\\
f^{(n)}(x) &= e^x & \Rightarrow && f^{(n)}(0) &= 1
\end{aligned}$$

Figure 9.9.4: The derivatives of $f(x) = e^x$ evaluated at $x = 0$.

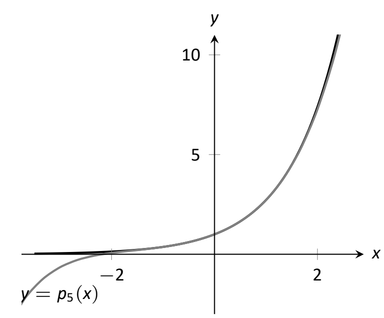


Figure 9.9.5: A plot of $f(x) = e^x$ and its 5^{th} degree Maclaurin polynomial $p_5(x)$.

Example 9.9.2 Finding and using Taylor polynomials

1. Find the n^{th} Taylor polynomial of $y = \ln x$ at $x = 1$.
2. Use $p_6(x)$ to approximate the value of $\ln 1.5$.
3. Use $p_6(x)$ to approximate the value of $\ln 2$.

SOLUTION

1. We begin by creating a table of derivatives of $\ln x$ evaluated at $x = 1$. While this is not as straightforward as it was in the previous example, a pattern does emerge, as shown in Figure 9.9.6.

$$\begin{aligned}
f(x) &= \ln x & \Rightarrow && f(1) &= 0\\
f'(x) &= 1/x & \Rightarrow && f'(1) &= 1\\
f''(x) &= -1/x^2 & \Rightarrow && f''(1) &= -1\\
f'''(x) &= 2/x^3 & \Rightarrow && f'''(1) &= 2\\
f^{(4)}(x) &= -6/x^4 & \Rightarrow && f^{(4)}(1) &= -6\\
&\vdots &&& &\vdots\\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{x^n} & \Rightarrow && f^{(n)}(1) &= (-1)^{n+1}(n-1)!
\end{aligned}$$

Figure 9.9.6: Derivatives of $\ln x$ evaluated at $x = 1$.

Notes:

Using Definition 9.9.1, we have

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}(x-1)^k.$$

2. We can compute $p_6(x)$ using our work above:

$$p_6(x) = (x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\frac{1}{5}(x-1)^5-\frac{1}{6}(x-1)^6.$$

Since $p_6(x)$ approximates $\ln x$ well near $x = 1$, we approximate $\ln 1.5 \approx p_6(1.5)$:

$$\begin{aligned} p_6(1.5) &= (1.5-1) - \frac{1}{2}(1.5-1)^2 + \frac{1}{3}(1.5-1)^3 \\ &\qquad - \frac{1}{4}(1.5-1)^4 + \frac{1}{5}(1.5-1)^5 - \frac{1}{6}(1.5-1)^6 \\ &= \frac{259}{640} \\ &\approx 0.404688. \end{aligned}$$

This is a good approximation as a calculator shows that $\ln 1.5 \approx 0.4055$. Figure 9.9.7 plots $y = \ln x$ with $y = p_6(x)$. We can see that $\ln 1.5 \approx p_6(1.5)$.

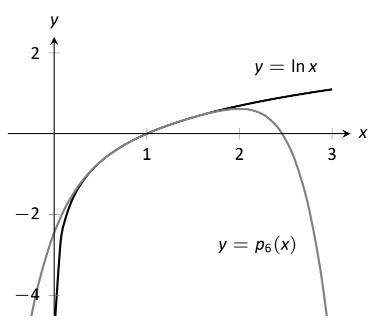


Figure 9.9.7: A plot of $y = \ln x$ and its 6th degree Taylor polynomial at $x = 1$.

3. We approximate $\ln 2$ with $p_6(2)$:

$$\begin{aligned} p_6(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 \\ &\qquad - \frac{1}{4}(2-1)^4 + \frac{1}{5}(2-1)^5 - \frac{1}{6}(2-1)^6 \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \\ &= \frac{37}{60} \\ &\approx 0.616667. \end{aligned}$$

This approximation is not terribly impressive: a hand held calculator shows that $\ln 2 \approx 0.693147$. The graph in Figure 9.9.7 shows that $p_6(x)$ provides less accurate approximations of $\ln x$ as x gets close to 0 or 2.

Surprisingly enough, even the 20th degree Taylor polynomial fails to approximate $\ln x$ for $x > 2$, as shown in Figure 9.9.8. We'll soon discuss why this is.

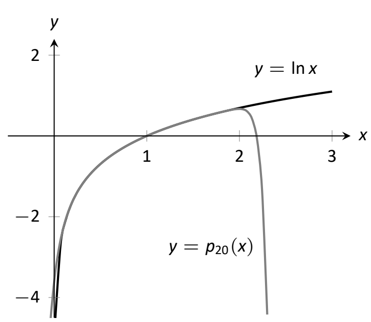


Figure 9.9.8: A plot of $y = \ln x$ and its 20th degree Taylor polynomial at $x = 1$.

Notes:

Taylor polynomials are used to approximate functions $f(x)$ in mainly two situations:

1. When $f(x)$ is known, but perhaps "hard" to compute directly. For instance, we can define $y = \cos x$ as either the ratio of sides of a right triangle ("adjacent over hypotenuse") or with the unit circle. However, neither of these provides a convenient way of computing $\cos 2$. A Taylor polynomial of sufficiently high degree can provide a reasonable method of computing such values using only operations usually hard-wired into a computer ($+$, $-$, $\times$ and $\div$).

2. When $f(x)$ is not known, but information about its derivatives is known. This occurs more often than one might think, especially in the study of differential equations.

In both situations, a critical piece of information to have is "How good is my approximation?" If we use a Taylor polynomial to compute $\cos 2$, how do we know how accurate the approximation is?

We had the same problem with Numerical Integration. Theorem 8.7.1 provided bounds on the error when using, say, Simpson's Rule to approximate a definite integral. These bounds allowed us to determine that, for example, using 10 subintervals provided an approximation within $\pm.01$ of the exact value. The following theorem gives similar bounds for Taylor (and hence Maclaurin) polynomials.

Note: Even though Taylor polynomials *could* be used in calculators and computers to calculate values of trigonometric functions, in practice they generally aren't. Other more efficient and accurate methods have been developed, such as the CORDIC algorithm.

Theorem 9.9.1 Taylor's Theorem

1. Let f be a function whose $(n+1)^{\text{th}}$ derivative exists on an open interval I and let c be in I. Then, for each x in I, there exists z_x between x and c such that

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \frac{f^{(n+1)}(z_x)}{(n+1)!}(x-c)^{n+1}.$$

2. $|R_n(x)| \leq \dfrac{\max_z \left|f^{(n+1)}(z)\right|}{(n+1)!}|x-c|^{n+1}$, where z is between x and c.

The first part of Taylor's Theorem states that $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is the n^{th} order Taylor polynomial and $R_n(x)$ is the remainder, or error, in the Taylor approximation. The second part gives bounds on how big that error can be. If the $(n+1)^{\text{th}}$ derivative is large, the error may be large; if x is far from

Notes:

c, the error may also be large. However, the $(n+1)!$ term in the denominator tends to ensure that the error gets smaller as n increases.

The following example computes error estimates for the approximations of $\ln 1.5$ and $\ln 2$ made in Example 9.9.2.

Example 9.9.3 Finding error bounds of a Taylor polynomial

Use Theorem 9.9.1 to find error bounds when approximating $\ln 1.5$ and $\ln 2$ with $p_6(x)$, the Taylor polynomial of degree 6 of $f(x) = \ln x$ at $x = 1$, as calculated in Example 9.9.2.

Solution

1. We start with the approximation of $\ln 1.5$ with $p_6(1.5)$. Taylor's Theorem references $\max \left|f^{(n+1)}(z)\right|$. In our situation, this is asking "How big can the 7^{th} derivative of $y = \ln x$ be on the interval $[1, 1.5]$?" The seventh derivative is $y = 6!/x^7$. The largest absolute value it attains on I is 720. Thus we can bound the error as:

$$\begin{aligned}|R_6(1.5)| &\leq \frac{\max \left|f^{(7)}(z)\right|}{7!}\left|1.5-1\right|^7\\ &\leq \frac{720}{5040}\cdot\frac{1}{2^7}\\ &\approx 0.001.\end{aligned}$$

We computed $p_6(1.5) = 0.404688$; using a calculator, we find $\ln 1.5 \approx 0.405465$, so the actual error is about 0.000778 (or 0.2%), which is less than our bound of 0.001. This affirms Taylor's Theorem; the theorem states that our approximation would be within about one thousandth of the actual value, whereas the approximation was actually closer.

2. The maximum value of the seventh derivative of f on $[1, 2]$ is again 720 (as the largest values come at $x = 1$). Thus

$$\begin{aligned}|R_6(2)| &\leq \frac{\max \left|f^{(7)}(z)\right|}{7!}\left|2-1\right|^7\\ &\leq \frac{720}{5040}\cdot 1^7\\ &\approx 0.15.\end{aligned}$$

This bound is not as nearly as good as before. Using the degree 6 Taylor polynomial at $x = 1$ will bring us within 0.15 of the correct answer. As $p_6(2) \approx 0.61667$, our error estimate guarantees that the actual value of $\ln 2$ is somewhere between 0.46 and 0.76. These bounds are not particularly useful.

Notes:

In reality, our approximation was only off by about 0.07 (or 11%). However, we are approximating ostensibly because we do not know the real answer. In order to be assured that we have a good approximation, we would have to resort to using a polynomial of higher degree.

We practice again. This time, we use Taylor's theorem to find n that guarantees our approximation is within a certain amount.

Example 9.9.4 Finding sufficiently accurate Taylor polynomials

Find n such that the n^{th} Taylor polynomial of $f(x) = \cos x$ at $x = 0$ approximates $\cos 2$ to within 0.001 of the actual answer. What is $p_n(2)$?

Solution Following Taylor's theorem, we need bounds on the size of the derivatives of $f(x) = \cos x$. In the case of this trigonometric function, this is easy. All derivatives of cosine are $\pm \sin x$ or $\pm \cos x$. In all cases, these functions are never greater than 1 in absolute value. We want the error to be less than 0.001. To find the appropriate n, consider the following inequalities:

$$\begin{aligned} \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|2-0\right|^{n+1} &\leq 0.001 \\ \frac{1}{(n+1)!}\cdot 2^{n+1} &\leq 0.001 \end{aligned}$$

We find an n that satisfies this last inequality with trial-and-error. When $n = 8$, we have $\dfrac{2^{8+1}}{(8+1)!} \approx 0.0014$; when $n = 9$, we have $\dfrac{2^{9+1}}{(9+1)!} \approx 0.000282 < 0.001$. Thus we want to approximate $\cos 2$ with $p_9(2)$.

We now set out to compute $p_9(x)$. We again need a table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$. A table of these values is given in Figure 9.9.9. Notice how the derivatives, evaluated at $x = 0$, follow a certain pattern. All the odd powers of x in the Taylor polynomial will disappear as their coefficient is 0. While our error bounds state that we need $p_9(x)$, our work shows that this will be the same as $p_8(x)$.

Since we are forming our polynomial at $x = 0$, we are creating a Maclaurin polynomial, and:

$$p_8(x) = \sum_{k=0}^{8} \frac{f^{(k)}(0)}{k!}x^k = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

We finally approximate $\cos 2$:

$$\cos 2 \approx p_8(2) = -\frac{131}{315} \approx -0.41587.$$

$$\begin{aligned}
f(x) &= \cos x &\Rightarrow&& f(0) &= 1 \\
f'(x) &= -\sin x &\Rightarrow&& f'(0) &= 0 \\
f''(x) &= -\cos x &\Rightarrow&& f''(0) &= -1 \\
f'''(x) &= \sin x &\Rightarrow&& f'''(0) &= 0 \\
f^{(4)}(x) &= \cos x &\Rightarrow&& f^{(4)}(0) &= 1 \\
f^{(5)}(x) &= -\sin x &\Rightarrow&& f^{(5)}(0) &= 0 \\
f^{(6)}(x) &= -\cos x &\Rightarrow&& f^{(6)}(0) &= -1 \\
f^{(7)}(x) &= \sin x &\Rightarrow&& f^{(7)}(0) &= 0 \\
f^{(8)}(x) &= \cos x &\Rightarrow&& f^{(8)}(0) &= 1 \\
f^{(9)}(x) &= -\sin x &\Rightarrow&& f^{(9)}(0) &= 0
\end{aligned}$$

Figure 9.9.9: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Notes:

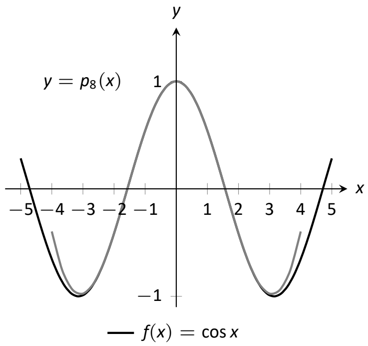


Figure 9.9.10: A graph of $f(x) = \cos x$ and its degree 8 Maclaurin polynomial.

Our error bound guarantees that this approximation is within 0.001 of the correct answer. Technology shows us that our approximation is actually within about 0.0003 (or 0.07%) of the correct answer.

Figure 9.9.10 shows a graph of $y = p_8(x)$ and $y = \cos x$. Note how well the two functions agree on about $(-\pi, \pi)$.

Example 9.9.5 Finding and using Taylor polynomials

1. Find the degree 4 Taylor polynomial, $p_4(x)$, for $f(x) = \sqrt{x}$ at $x = 4$.
2. Use $p_4(x)$ to approximate $\sqrt{3}$.
3. Find bounds on the error when approximating $\sqrt{3}$ with $p_4(3)$.

Solution

1. We begin by evaluating the derivatives of f at $x = 4$. This is done in Figure 9.9.11. These values allow us to form the Taylor polynomial $p_4(x)$:

$$p_4(x) =$$
$$2 + \frac{1}{4}(x-4) + \frac{-1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 + \frac{-15/2048}{4!}(x-4)^4.$$

2. As $p_4(x) \approx \sqrt{x}$ near $x = 4$, we approximate $\sqrt{3}$ with $p_4(3) = 1.73212$.
3. The largest value the fifth derivative of $f(x) = \sqrt{x}$ takes on $[3, 4]$ is when $x = 3$, at about 0.0234. Thus

$$|R_4(3)| \leq \frac{0.0234}{5!}\,|3-4|^5 \approx 0.00019.$$

This shows our approximation is accurate to at least the first 2 places after the decimal. It turns out that our approximation has an error of 0.00007, or 0.004%. A graph of $f(x) = \sqrt{x}$ and $p_4(x)$ is given in Figure 9.9.12. Note how the two functions are nearly indistinguishable on $(2, 7)$.

$$\begin{aligned}
f(x) &= \sqrt{x} & &\Rightarrow & f(4) &= 2\\
f'(x) &= \frac{1}{2\sqrt{x}} & &\Rightarrow & f'(4) &= \frac{1}{4}\\
f''(x) &= \frac{-1}{4x^{3/2}} & &\Rightarrow & f''(4) &= \frac{-1}{32}\\
f'''(x) &= \frac{3}{8x^{5/2}} & &\Rightarrow & f'''(4) &= \frac{3}{256}\\
f^{(4)}(x) &= \frac{-15}{16x^{7/2}} & &\Rightarrow & f^{(4)}(4) &= \frac{-15}{2048}
\end{aligned}$$

Figure 9.9.11: A table of the derivatives of $f(x) = \sqrt{x}$ evaluated at $x = 4$.

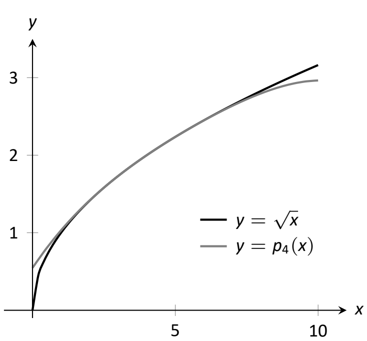


Figure 9.9.12: A graph of $f(x) = \sqrt{x}$ and its degree 4 Taylor polynomial at $x = 4$.

Most of this chapter has been devoted to the study of infinite series. This section has stepped aside from this study, focusing instead on finite summation of terms. In the next section, we will combine power series and Taylor polynomials into **Taylor Series**, where we represent a function with an infinite series.

Notes:

Exercises 9.9

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Maclaurin polynomial?
2. T/F: In general, $p_n(x)$ approximates $f(x)$ better and better as n gets larger.
3. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $p_2(x)$?
4. For some function $f(x)$, the Maclaurin polynomial of degree 4 is $p_4(x) = 6 + 3x - 4x^2 + 5x^3 - 7x^4$. What is $f'''(0)$?

Problems

In Exercises 5–12, find the Maclaurin polynomial of degree n for the given function.

5. $f(x) = e^{-x}, \quad n = 3$
6. $f(x) = \sin x, \quad n = 8$
7. $f(x) = x \cdot e^x, \quad n = 5$
8. $f(x) = \tan x, \quad n = 6$
9. $f(x) = e^{2x}, \quad n = 4$
10. $f(x) = \dfrac{1}{1-x}, \quad n = 4$
11. $f(x) = \dfrac{1}{1+x}, \quad n = 4$
12. $f(x) = \dfrac{1}{1+x}, \quad n = 7$

In Exercises 13–20, find the Taylor polynomial of degree n, at $x = c$, for the given function.

13. $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$
14. $f(x) = \ln(x+1), \quad n = 4, \quad c = 1$
15. $f(x) = \cos x, \quad n = 6, \quad c = \pi/4$
16. $f(x) = \sin x, \quad n = 5, \quad c = \pi/6$
17. $f(x) = \dfrac{1}{x}, \quad n = 5, \quad c = 2$
18. $f(x) = \dfrac{1}{x^2}, \quad n = 8, \quad c = 1$
19. $f(x) = \dfrac{1}{x^2+1}, \quad n = 3, \quad c = -1$
20. $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

In Exercises 21–24, approximate the function value with the indicated Taylor polynomial and give approximate bounds on the error.

21. Approximate $\sin 0.1$ with the Maclaurin polynomial of degree 3.
22. Approximate $\cos 1$ with the Maclaurin polynomial of degree 4.
23. Approximate $\sqrt{10}$ with the Taylor polynomial of degree 2 centered at $x = 9$.
24. Approximate $\ln 1.5$ with the Taylor polynomial of degree 3 centered at $x = 1$.

Exercises 25–28 ask for an n to be found such that $p_n(x)$ approximates $f(x)$ within a certain bound of accuracy.

25. Find n such that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.
26. Find n such that the Taylor polynomial of degree n of $f(x) = \sqrt{x}$, centered at $x = 4$, approximates $\sqrt{3}$ within 0.0001 of the actual value.
27. Find n such that the Maclaurin polynomial of degree n of $f(x) = \cos x$ approximates $\cos \pi/3$ within 0.0001 of the actual value.
28. Find n such that the Maclaurin polynomial of degree n of $f(x) = \sin x$ approximates $\sin \pi$ within 0.0001 of the actual value.

In Exercises 29–34, find the x^n term of the indicated Taylor polynomial.

29. Find a formula for the x^n term of the Maclaurin polynomial for $f(x) = e^x$.
30. Find a formula for the x^n term of the Maclaurin polynomial for $f(x) = \cos x$.
31. Find a formula for the x^n term of the Maclaurin polynomial for $f(x) = \sin x$.
32. Find a formula for the x^n term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1-x}$.
33. Find a formula for the x^n term of the Maclaurin polynomial for $f(x) = \dfrac{1}{1+x}$.
34. Find a formula for the x^n term of the Taylor polynomial for $f(x) = \ln x$ centered at $c = 1$.

In Exercises 35–36, approximate the solution to the given differential equation with a degree 4 Maclaurin polynomial.

35. $y' = y; \qquad y(0) = 1$
36. $y' = \dfrac{2}{y}; \qquad y(0) = 1$

9.10 Taylor Series

In Section 9.8, we showed how certain functions can be represented by a power series function. In Section 9.9, we showed how we can approximate functions with polynomials, given that enough derivative information is available. In this section we combine these concepts: if a function $f(x)$ is infinitely differentiable, we show how to represent it with a power series function.

Definition 9.10.1 Taylor and Maclaurin Series
Let $f(x)$ have derivatives of all orders at $x = c$.

1. The **Taylor Series of** $f(x)$**, centered at** c is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

2. Setting $c = 0$ gives the **Maclaurin Series of** $f(x)$:
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Watch the video:
Taylor and Maclaurin Series — Example 2 at
`https://youtu.be/0s8OtXFBLkY`

The difference between a Taylor polynomial and a Taylor series is the former is a polynomial, containing only a finite number of terms, whereas the latter is a series, a summation of an infinite set of terms. When creating the Taylor polynomial of degree n for a function $f(x)$ at $x = c$, we needed to evaluate f, and the first n derivatives of f, at $x = c$. When creating the Taylor series of f, we need to find a pattern that describes the n^{th} derivative of f at $x = c$. We demonstrate this in the next two examples.

Notes:

Example 9.10.1 The Maclaurin series of $f(x) = \cos x$

Find the Maclaurin series of $f(x) = \cos x$.

Solution In Example 9.9.4 we found the 8^{th} degree Maclaurin polynomial of $\cos x$. In doing so, we created the table shown in Figure 9.10.1. Notice how $f^{(n)}(0) = 0$ when n is odd, $f^{(n)}(0) = 1$ when n is divisible by 4, and $f^{(n)}(0) = -1$ when n is even but not divisible by 4. Thus the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

We can go further and write this as a summation. Since we only need the terms where the power of x is even, we write the power series in terms of x^{2n}:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

$$\begin{array}{rcl} f(x) = \cos x & \Rightarrow & f(0) = 1 \\ f'(x) = -\sin x & \Rightarrow & f'(0) = 0 \\ f''(x) = -\cos x & \Rightarrow & f''(0) = -1 \\ f'''(x) = \sin x & \Rightarrow & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & \Rightarrow & f^{(4)}(0) = 1 \\ f^{(5)}(x) = -\sin x & \Rightarrow & f^{(5)}(0) = 0 \\ f^{(6)}(x) = -\cos x & \Rightarrow & f^{(6)}(0) = -1 \\ f^{(7)}(x) = \sin x & \Rightarrow & f^{(7)}(0) = 0 \\ f^{(8)}(x) = \cos x & \Rightarrow & f^{(8)}(0) = 1 \\ f^{(9)}(x) = -\sin x & \Rightarrow & f^{(9)}(0) = 0 \end{array}$$

Figure 9.10.1: A table of the derivatives of $f(x) = \cos x$ evaluated at $x = 0$.

Example 9.10.2 The Taylor series of $f(x) = \ln x$ at $x = 1$

Find the Taylor series of $f(x) = \ln x$ centered at $x = 1$.

Solution Figure 9.10.2 shows the n^{th} derivative of $\ln x$ evaluated at $x = 1$ for $n = 0, \ldots, 5$, along with an expression for the n^{th} term:

$$f^{(n)}(1) = (-1)^{n+1}(n-1)! \quad \text{for } n \geq 1.$$

Remember that this is what distinguishes Taylor series from Taylor polynomials; we are very interested in finding a pattern for the n^{th} term, not just finding a finite set of coefficients for a polynomial. Since $f(1) = \ln 1 = 0$, we skip the first term and start the summation with $n = 1$, giving the Taylor series for $\ln x$, centered at $x = 1$, as

$$\sum_{n=1}^{\infty} (-1)^{n+1}(n-1)!\frac{1}{n!}(x-1)^n = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{(x-1)^n}{n}.$$

$$\begin{array}{rcl} f(x) = \ln x & \Rightarrow & f(1) = 0 \\ f'(x) = 1/x & \Rightarrow & f'(1) = 1 \\ f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1 \\ f'''(x) = 2/x^3 & \Rightarrow & f'''(1) = 2 \\ f^{(4)}(x) = -6/x^4 & \Rightarrow & f^{(4)}(1) = -6 \\ f^{(5)}(x) = 24/x^5 & \Rightarrow & f^{(5)}(1) = 24 \\ \vdots & & \vdots \\ f^{(n)}(x) = & \Rightarrow & f^{(n)}(1) = \\ \frac{(-1)^{n+1}(n-1)!}{x^n} & & (-1)^{n+1}(n-1)! \end{array}$$

Figure 9.10.2: Derivatives of $\ln x$ evaluated at $x = 1$.

It is important to note that Definition 9.10.1 defines a Taylor series given a function $f(x)$; however, we *cannot* yet state that $f(x)$ *is equal* to its Taylor series. We will find that "most of the time" they are equal, but we need to consider the conditions that allow us to conclude this.

Theorem 9.9.1 states that the error between a function and its n^{th}-degree Taylor polynomial is $R_n(x)$, where

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x-c|^{n+1}.$$

Notes:

If $R_n(x)$ goes to 0 for each x in an interval I as n approaches infinity, we conclude that the function is equal to its Taylor series expansion.

Theorem 9.10.1 Function and Taylor Series Equality
Let $f(x)$ have derivatives of all orders at $x = c$, let $R_n(x)$ be as stated in Theorem 9.9.1, and let I be an interval on which the Taylor series of $f(x)$ converges. If $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x in I, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \text{ on } I.$$

We demonstrate the use of this theorem in an example.

Example 9.10.3 Establishing equality of a function and its Taylor series
Show that, for all x, $f(x) = \cos x$ is equal to its Maclaurin series as found in Example 9.10.1.

Solution Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}|x|^{n+1}.$$

Since all derivatives of $\cos x$ are $\pm\sin x$ or $\pm\cos x$, whose magnitudes are bounded by 1, we can state

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

which implies

$$-\frac{|x|^{n+1}}{(n+1)!} \leq R_n(x) \leq \frac{|x|^{n+1}}{(n+1)!}.$$

For any x, $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Applying the Squeeze Theorem to our last inequality, we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all } x.$$

It is natural to assume that a function is equal to its Taylor series on the series' interval of convergence, but this is not necessarily the case. In order to properly

Notes:

establish equality, one must use Theorem 9.10.1. This is a bit disappointing, as we developed beautiful techniques for determining the interval of convergence of a power series, and proving that $R_n(x) \to 0$ can be cumbersome as it deals with high order derivatives of the function.

There is good news. A function $f(x)$ that is equal to its Taylor series, centered at any point the domain of $f(x)$, is said to be an **analytic function**, and most, if not all, functions that we encounter within this course are analytic functions. Generally speaking, any function that one creates with elementary functions (polynomials, exponentials, trigonometric functions, etc.) that is not piecewise defined is probably analytic. For most functions, we assume the function is equal to its Taylor series on the series' interval of convergence and only use Theorem 9.10.1 when we suspect something may not work as expected. The converse is also true: if a function is equal to *some* power series on an interval, then that power series is the Taylor series of the function.

We develop the Taylor series for one more important function, then give a table of the Taylor series for a number of common functions.

Example 9.10.4 The Binomial Series

Find the Maclaurin series of $f(x) = (1+x)^k$, $k \neq 0$.

Solution When k is a positive integer, the Maclaurin series is finite. For instance, when $k = 4$, we have

$$f(x) = (1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The coefficients of x when k is a positive integer are known as the *binomial coefficients*, giving the series we are developing its name.

When $k = 1/2$, we have $f(x) = \sqrt{1+x}$. Knowing a series representation of this function would give a useful way of approximating $\sqrt{1.3}$, for instance.

To develop the Maclaurin series for $f(x) = (1+x)^k$ for any value of $k \neq 0$, we consider the derivatives of f evaluated at $x = 0$:

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1\\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k\\
f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1)\\
f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= k(k-1)\cdots\big(k-(n-1)\big)(1+x)^{k-n} & &\vdots\\
& & f^{(n)}(0) &= k(k-1)\cdots\big(k-(n-1)\big)
\end{aligned}$$

Notes:

Thus the Maclaurin series for $f(x) = (1+x)^k$ is

$$1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots+\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n+\cdots$$

It is important to determine the interval of convergence of this series. With

$$a_n = \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n,$$

we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty}\frac{\left|\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}\right|}{\left|\frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!}x^n\right|}\\
&= \lim_{n\to\infty}\left|\frac{k-n}{n+1}x\right|\\
&= |x|.
\end{aligned}$$

The series converges absolutely when the limit of the Ratio Test is less than 1; therefore, we have absolute convergence when $|x| < 1$.

While outside the scope of this text, the interval of convergence depends on the value of k. When $k > 0$, the interval of convergence is $[-1, 1]$. When $-1 < k < 0$, the interval of convergence is $(-1, 1]$. If $k \leq -1$, the interval of convergence is $(-1, 1)$.

We learned that Taylor polynomials offer a way of approximating a "difficult to compute" function with a polynomial. Taylor series offer a way of exactly representing a function with a series. One probably can see the use of a good approximation; is there any use of representing a function exactly as a series?

While we appreciate the mathematical beauty of Taylor series (which is reason enough to study them), there are practical uses as well. They provide a valuable tool for solving a variety of problems, including problems relating to integration and differential equations.

In Key Idea 9.10.1 (on the following page) we give a table of the Maclaurin series of a number of common functions. We then give a theorem about the "algebra of power series," that is, how we can combine power series to create power series of new functions. This allows us to find the Taylor series of functions like $f(x) = e^x \cos x$ by knowing the Taylor series of e^x and $\cos x$.

Before we investigate combining functions, consider the Taylor series for the arctangent function (see Key Idea 9.10.1). Knowing that $\tan^{-1}(1) = \pi/4$, we

Notes:

can use this series to approximate the value of π:

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$
$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

Unfortunately, this particular expansion of π converges very slowly. The first 100 terms approximate π as 3.13159, which is not particularly good.

Key Idea 9.10.1 Important Maclaurin Series Expansions

Function and Series	**First Few Terms**	**Interval of Convergence**
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty, \infty)$
$\ln(x+1) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$(-1, 1]$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$1 + x + x^2 + x^3 + \cdots$	$(-1, 1)$
$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots\big(k-(n-1)\big)}{n!} x^n$	$1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots$	$\begin{cases} (-1,1) & k \leq -1 \\ (-1,1] & -1 < k < 0 \\ [-1,1] & 0 < k \end{cases}$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1, 1]$

Notes:

Theorem 9.10.2 Algebra of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and let $h(x)$ be continuous.

1. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)x^n \quad$ for $|x| < R$.
2. $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) =$
   $\sum_{n=0}^{\infty} \big(a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0\big)x^n$ for $|x| < R$.
3. $f\big(h(x)\big) = \sum_{n=0}^{\infty} a_n \big(h(x)\big)^n \quad$ for $|h(x)| < R$.

Example 9.10.5 Combining Taylor series

Write out the first 3 terms of the Maclaurin Series for $f(x) = e^x \cos x$ using Key Idea 9.10.1 and Theorem 9.10.2.

Solution Key Idea 9.10.1 informs us that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{and} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots .$$

Applying Theorem 9.10.2, we find that

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right).$$

Distribute the right hand expression across the left:

$$\begin{aligned} &= 1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^2}{2!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{x^3}{3!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\ &\quad + \frac{x^4}{4!}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \cdots \end{aligned}$$

Notes:

Distribute again and collect like terms.

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} + \cdots$$

While this process is a bit tedious, it is much faster than evaluating all the necessary derivatives of $e^x \cos x$ and computing the Taylor series directly.

Because the series for e^x and $\cos x$ both converge on $(-\infty, \infty)$, so does the series expansion for $e^x \cos x$.

Example 9.10.6 Creating new Taylor series

Use Theorem 9.10.2 to create the Taylor series for $y = \sin(x^2)$ centered at $x = 0$ and a series for $y = \ln(\sqrt{x})$ centered at $c = 1$.

Solution Given that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we simply substitute x^2 for x in the series, giving

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} \cdots.$$

Since the Taylor series for $\sin x$ has an infinite radius of convergence, so does the Taylor series for $\sin(x^2)$.

The Taylor expansion for $\ln(x+1)$ given in Key Idea 9.10.1 is centered at $x = 0$, so we can center the series for $\ln(\sqrt{x})$ at $x = 1$. With

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots,$$

we substitute $\sqrt{x}$ for x to obtain

$$\ln(\sqrt{x}) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\sqrt{x}-1)^n}{n} = (\sqrt{x}-1) - \frac{(\sqrt{x}-1)^2}{2} + \frac{(\sqrt{x}-1)^3}{3} - \cdots.$$

While this is not strictly a power series because of the $\sqrt{x}$, it is a series that allows us to study the function $\ln(\sqrt{x})$. Since the interval of convergence of $\ln x$ is $(0, 2]$, and the range of $\sqrt{x}$ on $(0, 4]$ is $(0, 2]$, the interval of convergence of this series expansion of $\ln(\sqrt{x})$ is $(0, 4]$.

Note: In Example 9.10.6, one could create a series for $\ln(\sqrt{x})$ by simply recognizing that $\ln(\sqrt{x}) = \ln(x^{1/2}) = 1/2 \ln x$, and hence multiplying the Taylor series for $\ln x$ by $1/2$. This example was chosen to demonstrate other aspects of series, such as the fact that the interval of convergence changes.

Notes:

Example 9.10.7 Using Taylor series to evaluate definite integrals

Use the Taylor series of e^{-x^2} to evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution We learned, when studying Numerical Integration, that e^{-x^2} does not have an antiderivative expressible in terms of elementary functions. This means any definite integral of this function must have its value approximated, and not computed exactly.

We can quickly write out the Taylor series for e^{-x^2} using the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and so

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots . \end{aligned}$$

We use Theorem 9.8.3 to integrate:

$$\int e^{-x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

This *is* the antiderivative of e^{-x^2}; while we can write it out as a series, we cannot write it out in terms of elementary functions. We can evaluate the definite integral $\displaystyle\int_0^1 e^{-x^2}\,dx$ using this antiderivative; substituting 1 and 0 for x and subtracting gives

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \cdots .$$

Summing the 5 terms shown above gives the approximation of 0.74749. Since this is an alternating series, we can use the Alternating Series Approximation Theorem, (Theorem 9.5.3), to determine how accurate this approximation is. The next term of the series is $1/(11\cdot 5!) \approx 0.00075758$. Thus we know our approximation is within 0.00075758 of the actual value of the integral. This is arguably much less work than using Simpson's Rule to approximate the value of the integral.

Notes:

Another advantage to using Taylor series instead of Simpson's Rule is for making subsequent approximations. We found in Example 8.7.7 that the error in using Simpson's Rule for $\int_0^1 e^{-x^2}\,dx$ with four intervals was 0.00026. If we wanted to decrease that error, we would need to use more intervals, essentially starting the problem over. Using a Taylor series, if we wanted a more accurate approximation, we can just subtract the next term $1/(11\cdot 5!)$ to get an approximation of 0.7467, with an error of at most $1/(13\cdot 6!)\approx 0.0001$.

Finding a pattern in the coefficients that match the series expansion of a known function, such as those shown in Key Idea 9.10.1, can be difficult. What if the coefficients are given in their reduced form; how could we still recover the function?

Suppose that all we know is that

$$a_0=1,\quad a_1=2,\quad a_2=2,\quad a_3=\frac{4}{3},\quad a_4=\frac{2}{3}.$$

Definition 9.10.1 states that each term of the Taylor expansion of a function includes an $n!$. This allows us to say that

$$a_2=2=\frac{b_2}{2!},\quad a_3=\frac{4}{3}=\frac{b_3}{3!},\quad \text{and}\quad a_4=\frac{2}{3}=\frac{b_4}{4!}$$

for some values b_2, b_3 and b_4. Solving for these values, we see that $b_2=4$, $b_3=8$ and $b_4=16$. That is, we are recovering the pattern $b_n=2^n$, allowing us to write

$$\begin{aligned} f(x)=\sum_{n=0}^{\infty}a_nx^n&=\sum_{n=0}^{\infty}\frac{b_n}{n!}x^n\\ &=1+2x+\frac{4}{2!}x^2+\frac{8}{3!}x^3+\frac{16}{4!}x^4+\cdots \end{aligned}$$

From here it is easier to recognize that the series is describing an exponential function.

This chapter introduced sequences, which are ordered lists of numbers, followed by series, wherein we add up the terms of a sequence. We quickly saw that such sums do not always add up to "infinity," but rather converge. We studied tests for convergence, then ended the chapter with a formal way of defining functions based on series. Such "series-defined functions" are a valuable tool in solving a number of different problems throughout science and engineering.

Coming in the next chapters are new ways of defining curves in the plane apart from using functions of the form $y=f(x)$. Curves created by these new methods can be beautiful, useful, and important.

Notes:

Exercises 9.10

Terms and Concepts

1. What is the difference between a Taylor polynomial and a Taylor series?
2. What theorem must we use to show that a function is equal to its Taylor series?

Problems

Key Idea 9.10.1 gives the n^{th} term of the Taylor series of common functions. In Exercises 3–6, verify the formula given in the Key Idea by finding the first few terms of the Taylor series of the given function and identifying a pattern.

3. $f(x) = e^x; \quad c = 0$
4. $f(x) = \sin x; \quad c = 0$
5. $f(x) = 1/(1-x); \quad c = 0$
6. $f(x) = \tan^{-1} x; \quad c = 0$

In Exercises 7–12, find a formula for the n^{th} term of the Taylor series of $f(x)$, centered at c, by finding the coefficients of the first few powers of x and looking for a pattern. (The formulas for several of these are found in Key Idea 9.10.1; show work verifying these formula.)

7. $f(x) = \cos x; \quad c = \pi/2$
8. $f(x) = 1/x; \quad c = 1$
9. $f(x) = e^{-x}; \quad c = 0$
10. $f(x) = \ln(1+x); \quad c = 0$
11. $f(x) = x/(x+1); \quad c = 1$
12. $f(x) = \sin x; \quad c = \pi/4$

In Exercises 13–16, show that the Taylor series for $f(x)$, as given in Key Idea 9.10.1, is equal to $f(x)$ by applying Theorem 9.10.1; that is, show $\lim\limits_{n\to\infty} R_n(x) = 0$.

13. $f(x) = e^x$
14. $f(x) = \sin x$
15. $f(x) = \ln(x+1)$ (show equality only on $(0,1)$).
16. $f(x) = 1/(1-x)$ (show equality only on $(-1,0)$)

In Exercises 17–20, use the Taylor series given in Key Idea 9.10.1 to verify the given identity.

17. $\cos(-x) = \cos x$
18. $\sin(-x) = -\sin x$
19. $\frac{d}{dx}\big(\sin x\big) = \cos x$
20. $\frac{d}{dx}\big(\cos x\big) = -\sin x$

In Exercises 21–24, write out the first 5 terms of the Binomial series with the given k-value.

21. $k = 1/2$
22. $k = -1/2$
23. $k = 1/3$
24. $k = 4$

In Exercises 25–30, use the Taylor series given in Key Idea 9.10.1 to create the Taylor series of the given functions.

25. $f(x) = \cos\big(x^2\big)$
26. $f(x) = e^{-x}$
27. $f(x) = \sin\big(2x+3\big)$
28. $f(x) = \tan^{-1}\big(x/2\big)$
29. $f(x) = e^x \sin x$ (only find the first non-zero 4 terms)
30. $f(x) = (1+x)^{1/2}\cos x$ (only find the first non-zero 4 terms)

In Exercises 31–32, approximate the value of the given definite integral by using the first 4 nonzero terms of the integrand's Taylor series.

31. $\displaystyle\int_0^{\sqrt{\pi}} \sin\big(x^2\big)\, dx$
32. $\displaystyle\int_0^{\pi^2/4} \cos\big(\sqrt{x}\big)\, dx$

10.0 Chapter Prerequisites — Conic Sections

The material in this section provides a basic review of and practice problems for pre-calculus skills essential to your success in Calculus. You should take time to review this section and work the suggested problems (checking your answers against those in the back of the book). Since this content is a pre-requisite for Calculus, reviewing and mastering these skills are considered your responsibility. This means that minimal, and in some cases no, class time will be devoted to this section. When you identify areas that you need help with we strongly urge you to seek assistance outside of class from your instructor or other student tutoring service.

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip-to-tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 10.0.1. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

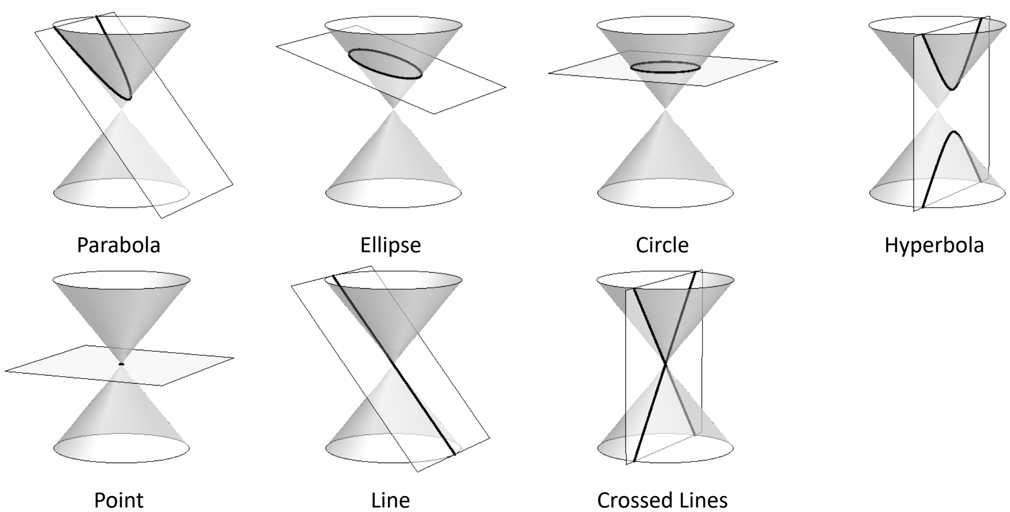


Figure 10.0.1: Conic Sections

When the plane does contain the origin, three **degenerate** cones can be formed as shown the bottom row of Figure 10.0.1: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the **locus**, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

Parabolas

Definition 10.0.1 Parabola
A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

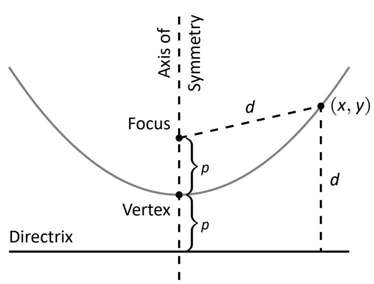


Figure 10.0.2: Illustrating the definition of the parabola and establishing an algebraic formula.

Figure 10.0.2 illustrates this definition. The point halfway between the focus and the directrix is the **vertex**. The line through the focus, perpendicular to the directrix, is the **axis of symmetry**, as the portion of the parabola on one side of this line is the mirror-image of the portion on the opposite side.

The geometric definition of the parabola and distance formula can be used to derive the quadratic function whose graph is a parabola with vertex at the origin.

$$y = \frac{1}{4p}x^2.$$

Applying transformations of functions we get the following standard form of the parabola.

Notes:

Key Idea 10.0.1 General Equation of a Parabola

1. **Vertical Axis of Symmetry:** The equation of the parabola with vertex at (h,k), directrix $y = k - p$, and focus at $(h, k+p)$ in standard form is
$$y = \frac{1}{4p}(x-h)^2 + k.$$
2. **Horizontal Axis of Symmetry:** The equation of the parabola with vertex at (h,k), directrix $x = h - p$, and focus at $(h+p, k)$ in standard form is
$$x = \frac{1}{4p}(y-k)^2 + h.$$

Note: p is not necessarily a positive number.

Example 10.0.1 Finding the equation of a parabola

Give the equation of the parabola with focus at $(1,2)$ and directrix at $y = 3$.

Solution The vertex is located halfway between the focus and directrix, so $(h,k) = (1, 2.5)$. This gives $p = -0.5$. Using Key Idea 10.0.1 we have the equation of the parabola as

$$y = \frac{1}{4(-0.5)}(x-1)^2 + 2.5 = -\frac{1}{2}(x-1)^2 + 2.5.$$

The parabola is sketched in Figure 10.0.3.

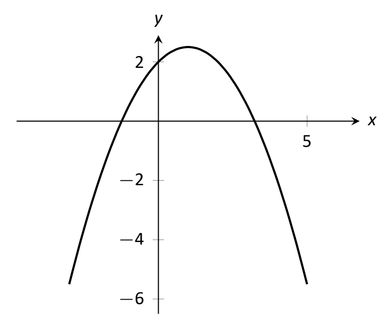


Figure 10.0.3: The parabola described in Example 10.0.1.

Ellipses

Definition 10.0.2 Ellipse

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 10.0.4.

We can again find an algebraic equation for an ellipse using this geometric definition.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

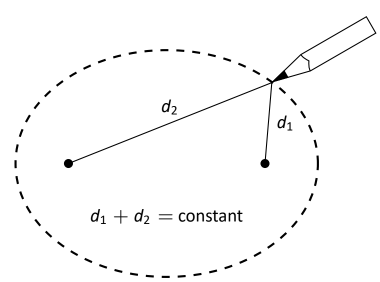


Figure 10.0.4: Illustrating the construction of an ellipse with pins, pencil and string.

Notes:

As shown in Figure 10.0.5, the values of a and b have meaning. In general, the two foci of an ellipse lie on the **major axis** of the ellipse, and the midpoint of the segment joining the two foci is the **center**. The major axis intersects the ellipse at two points, each of which is a **vertex**. The line segment through the center and perpendicular to the major axis is the **minor axis**. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

Allowing for the shifting of the ellipse gives the following standard equations.

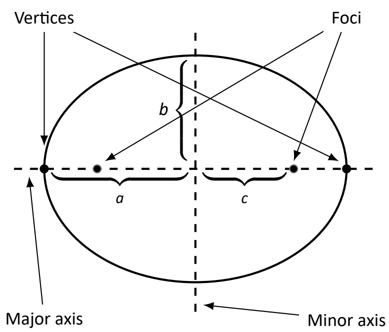


Figure 10.0.5: Labeling the significant features of an ellipse.

Key Idea 10.0.2 Standard Equation of the Ellipse

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. **Horizontal major axis:** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$
2. **Vertical major axis:** $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 10.0.2 Finding the equation of an ellipse

Find the general equation of the ellipse graphed in Figure 10.0.6.

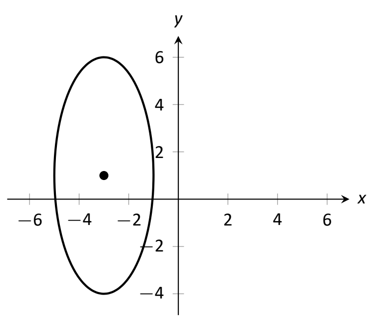


Figure 10.0.6: The ellipse used in Example 10.0.2.

Solution The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

Example 10.0.3 Graphing an ellipse

Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Solution It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2 + 9y^2 - 8x - 36y = -4 \quad \Rightarrow \quad (4x^2 - 8x) + (9y^2 - 36y) = -4.$$

Notes:

Now we complete the squares.

$$\begin{aligned}(4x^2-8x)+(9y^2-36y)&=-4\\ 4(x^2-2x)+9(y^2-4y)&=-4\\ 4(x^2-2x+1-1)+9(y^2-4y+4-4)&=-4\\ 4\big((x-1)^2-1\big)+9\big((y-2)^2-4\big)&=-4\\ 4(x-1)^2-4+9(y-2)^2-36&=-4\\ 4(x-1)^2+9(y-2)^2&=36\\ \frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}&=1.\end{aligned}$$

We see the center of the ellipse is at $(1,2)$. We have $a=3$ and $b=2$; the major axis is horizontal, so the vertices are located at $(-2,2)$ and $(4,2)$. We find $c=\sqrt{9-4}=\sqrt{5}\approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1\pm 2.24,2)$. This is all graphed in Figure 10.0.7.

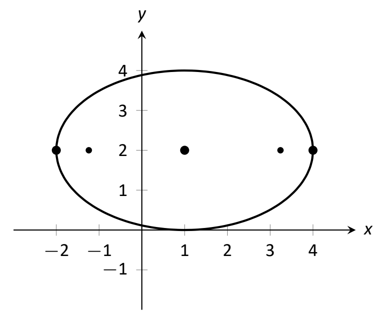


Figure 10.0.7: Graphing the ellipse in Example 10.0.3.

Hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 10.0.3 Hyperbola
A **hyperbola** is the locus of all points where the absolute value of the difference of distances from two fixed points, each a focus of the hyperbola, is constant.

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the **transverse axis** of the hyperbola; the midpoint of the line segment joining the foci is the **center** of the hyperbola. The transverse axis intersects the hyperbola at two points, each a **vertex** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis.** This is illustrated in Figure 10.0.8. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

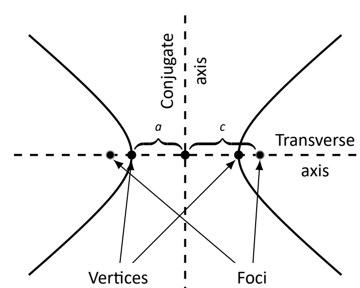


Figure 10.0.8: Labeling the significant features of a hyperbola.

Notes:

Key Idea 10.0.3 Standard Equation of a Hyperbola

The equation of a hyperbola centered at (h,k) in standard form is:

1. **Horizontal Transverse Axis:** $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$
2. **Vertical Transverse Axis:** $$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2=a^2+b^2$.

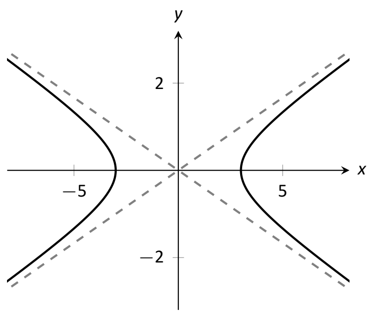


Figure 10.0.9: Graphing the hyperbola $\frac{x^2}{9}-\frac{y^2}{1}=1$ along with its asymptotes, $y=\pm x/3$.

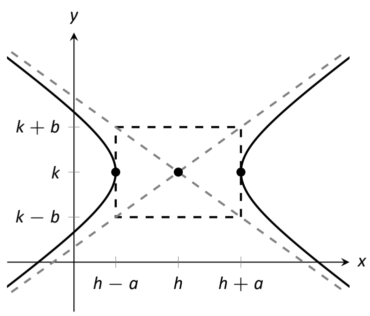


Figure 10.0.10: Using the asymptotes of a hyperbola as a graphing aid.

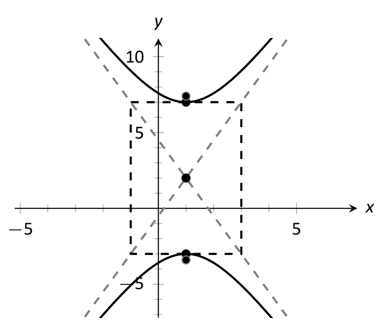


Figure 10.0.11: Graphing the hyperbola in Example 10.0.4.

Graphing Hyperbolas

Consider the hyperbola $\frac{x^2}{9}-\frac{y^2}{1}=1$. Solving for y, we find $y=\pm\sqrt{x^2/9-1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y\approx\pm\sqrt{x^2/9}=\pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y=\pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 10.0.9.

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h,k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 10.0.10 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h,k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

Horizontal Transverse Axis	Vertical Transverse Axis
$y=\pm\frac{b}{a}(x-h)+k$	$y=\pm\frac{a}{b}(x-h)+k.$

Example 10.0.4 Graphing a hyperbola

Sketch the hyperbola given by $\frac{(y-2)^2}{25}-\frac{(x-1)^2}{4}=1.$

Solution The hyperbola is centered at $(1,2)$; $a=5$ and $b=2$. In Figure 10.0.11 we draw the prescribed rectangle centered at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2=a^2+b^2$, we have $c=\sqrt{29}\approx 5.4$. Thus the foci are located at $(1,2\pm 5.4)$ as shown in the figure.

Notes:

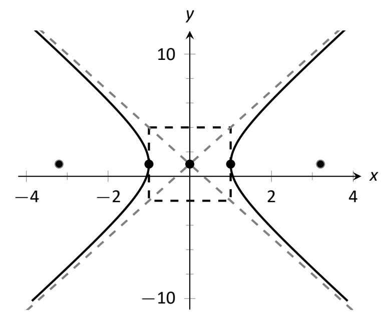


Figure 10.0.12: Graphing the hyperbola in Example 10.0.5.

Example 10.0.5 Graphing a hyperbola

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10$.

Solution We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned} 9x^2 - y^2 + 2y &= 10 \\ 9x^2 - (y^2 - 2y) &= 10 \\ 9x^2 - (y^2 - 2y + 1 - 1) &= 10 \\ 9x^2 - \big((y-1)^2 - 1\big) &= 10 \\ 9x^2 - (y-1)^2 &= 9 \\ x^2 - \frac{(y-1)^2}{9} &= 1 \end{aligned}$$

We see the hyperbola is centered at $(0, 1)$, with a horizontal transverse axis, where $a = 1$ and $b = 3$. The appropriate rectangle is sketched in Figure 10.0.12 along with the asymptotes of the hyperbola. The vertices are located at $(\pm 1, 1)$. We have $c = \sqrt{10} \approx 3.2$, so the foci are located at $(\pm 3.2, 1)$ as shown in Figure 10.0.12.

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

Notes:

Exercises 10.0

Problems

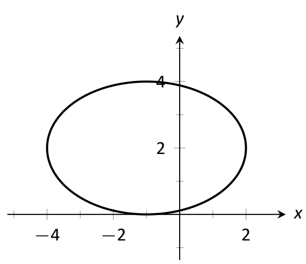
In Exercises 1–8, find the equation of the parabola defined by the given information. Sketch the parabola.

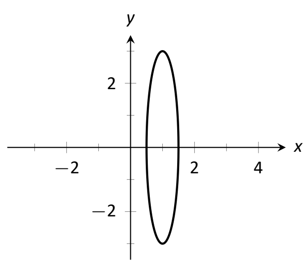
1. Focus: $(3, 2)$; directrix: $y = 1$
2. Focus: $(-1, -4)$; directrix: $y = 2$
3. Focus: $(1, 5)$; directrix: $x = 3$
4. Focus: $(1/4, 0)$; directrix: $x = -1/4$
5. Focus: $(1, 1)$; vertex: $(1, 2)$
6. Focus: $(-3, 0)$; vertex: $(0, 0)$
7. Vertex: $(0, 0)$; directrix: $y = -1/16$
8. Vertex: $(2, 3)$; directrix: $x = 4$

In Exercises 9–10, sketch the ellipse defined by the given equation. Label the center, foci and vertices.

9. $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{5} = 1$
10. $\dfrac{1}{25}x^2 + \dfrac{1}{9}(y+3)^2 = 1$

In Exercises 11–12, find the equation of the ellipse shown in the graph.

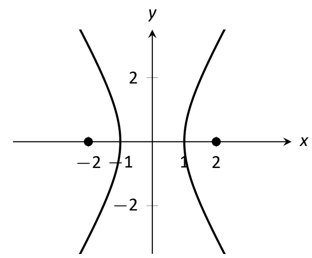
11. 

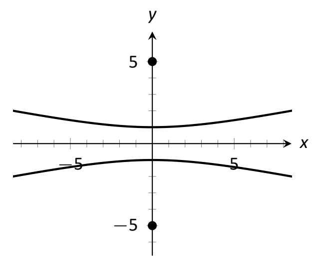
12. 

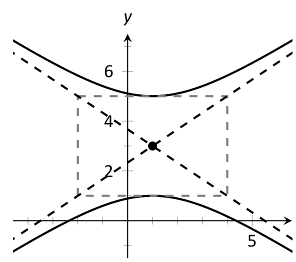
In Exercises 13–16, write the equation of the given ellipse in standard form.

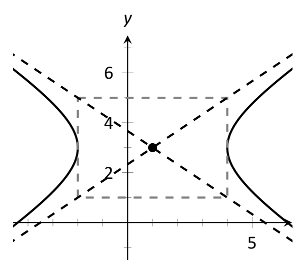
13. $x^2 - 2x + 2y^2 - 8y = -7$
14. $5x^2 + 3y^2 = 15$
15. $3x^2 + 2y^2 - 12y + 6 = 0$
16. $x^2 + y^2 - 4x - 4y + 4 = 0$

In Exercises 17–20, find the equation of the hyperbola shown in the graph.

17. 

18. 

19. 

20. 

In Exercises 21–22, sketch the hyperbola defined by the given equation. Label the center.

21. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$
22. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

In Exercises 23–26, write the equation of the hyperbola in standard form.

23. $3x^2 - 4y^2 = 12$
24. $3x^2 - y^2 + 2y = 10$
25. $x^2 - 10y^2 + 40y = 30$
26. $(4y - x)(4y + x) = 4$

10: Curves in the Plane

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important — and beautiful — new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

One aspect that we'll be interested in is "how long is this curve?" Before we explore that idea for these new ways to draw curves, we'll start by exploring how long a curve is when we've gotten it from a regular $y = f(x)$ function.

10.1 Arc Length and Surface Area

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?

2. Given a solid, what is its volume?

Notes:

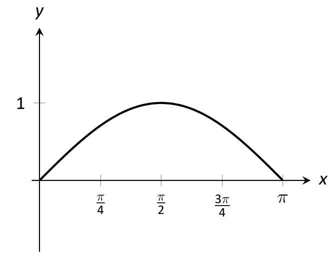
In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 10.1.1 (a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

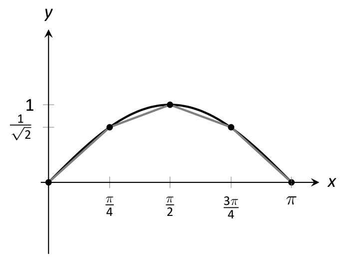
As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight-line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

In Figure 10.1.1 (b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equal length subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.



(a)



(b)

Figure 10.1.1: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_{i-1}, x_i]$.

Figure 10.1.2 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

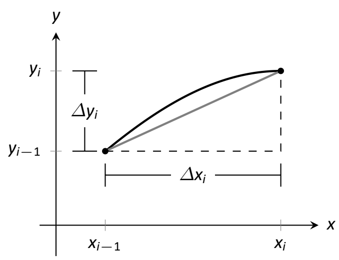


Figure 10.1.2: Zooming in on the i^{th} subinterval $[x_{i-1}, x_i]$ of a partition of $[a, b]$.

As it is written, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 \left(1 + \frac{(\Delta y_i)^2}{(\Delta x_i)^2}\right)}.$$

Now pull the $(\Delta x_i)^2$ term out of the square root:

$$L \approx \sum_{i=1}^{n} \sqrt{1 + \frac{(\Delta y_i)^2}{(\Delta x_i)^2}}\, \Delta x_i.$$

Notes:

This is nearly a Riemann Sum. Consider the $(\Delta y_i)^2/(\Delta x_i)^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$L \approx \sum_{i=1}^{n} \sqrt{1+[f'(c_i)]^2}\,\Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous on $[a,b]$, we can invoke Theorem 5.3.2 and conclude

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

Key Idea 10.1.1 Arc Length
Let f be differentiable on an open interval containing $[a,b]$, where f' is also continuous on $[a,b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

Watch the video:
Arc Length at
`https://youtu.be/PwmCZAWeRNE`

As the integrand contains a square root, it is often difficult to use the formula in Key Idea 10.1.1 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 10.1.1 Finding arc length
Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution A graph of f is given in Figure 10.1.3. We begin by finding

Notes:

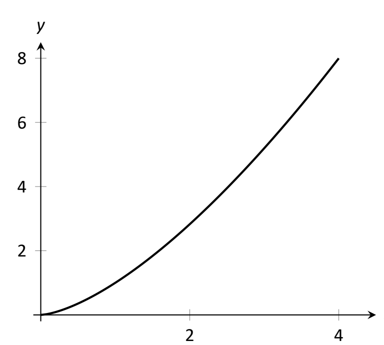


Figure 10.1.3: A graph of $f(x) = x^{3/2}$ from Example 10.1.1.

$f'(x) = \frac{3}{2}x^{1/2}$. Using the formula, we find the arc length L as

$$\begin{aligned}
L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\ \mathrm{d}x \\
&= \int_0^4 \sqrt{1+\frac{9}{4}x}\ \mathrm{d}x \\
&= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\ \mathrm{d}x \\
&= \frac{2}{3}\cdot\frac{4}{9}\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^4 \\
&= \frac{8}{27}\left(10^{3/2}-1\right)\text{ units.}
\end{aligned}$$

Example 10.1.2 Finding arc length

Find the arc length of $f(x) = \dfrac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

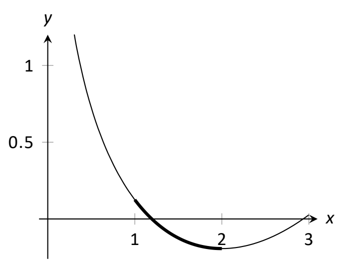


Figure 10.1.4: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 10.1.2.

Solution A graph of f is given in Figure 10.1.4; the portion of the curve measured in this problem is in bold. This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned}
L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\ \mathrm{d}x \\
&= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\ \mathrm{d}x \\
&= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\ \mathrm{d}x \\
&= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\ \mathrm{d}x \\
&= \int_1^2 \left(\frac{x}{4}+\frac{1}{x}\right)\ \mathrm{d}x \\
&= \left(\frac{x^2}{8}+\ln x\right)\Big|_1^2 \\
&= \frac{3}{8}+\ln 2\text{ units.}
\end{aligned}$$

Notes:

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 10.1.3 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution This is somewhat of a mathematical curiosity; Example 5.4.3 showed that the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 10.1.5 gives the integrand evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\begin{aligned}\int_0^\pi \sqrt{1+\cos^2 x}\,dx &\approx \frac{\pi - 0}{4\cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right)\\ &\approx 3.82918.\end{aligned}$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good. Our approximation of 3.79 from the beginning of this section isn't as close.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

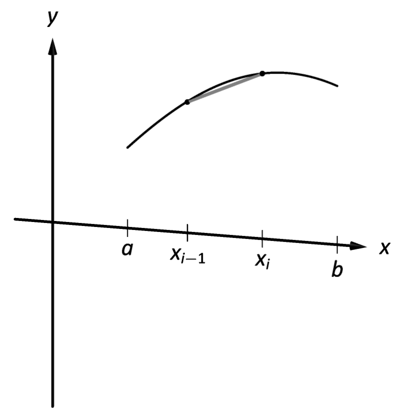
Figure 10.1.5: A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example 10.1.3.

Surface Area of Solids of Revolution

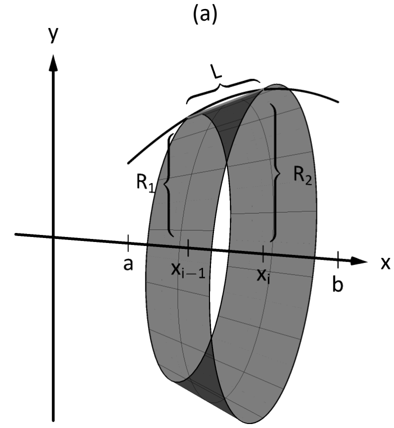
We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 10.1.6(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 10.1.6(b). The surface area of a frustum of a cone is

$$A = 2\pi r_{\text{avg}} L,$$



(a)



(b)

Figure 10.1.6: Establishing the formula for surface area.

Notes:

where r_{avg} is the average of R_1 and R_2. The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + [f'(c_i)]^2}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval: $f(x_{i-1})$ and $f(x_i)$. Thus the surface area of this sample frustum of the cone is approximately

$$2\pi\frac{f(x_{i-1}) + f(x_i)}{2}\sqrt{1 + [f'(c_i)]^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_{i-1}, x_i]$ such that $f(d_i) = \dfrac{f(x_{i-1}) + f(x_i)}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + [f'(c_i)]^2}\Delta x_i,$$

which is almost a Riemann Sum (we would need $d_i = c_i$ to remove the "almost"). Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the upcoming Key Idea.

If instead we revolve $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_{i-1} and x_i; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Key Idea 10.1.2.

Key Idea 10.1.2 Surface Area of a Solid of Revolution

Let f be differentiable on an open interval containing $[a, b]$ where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\,dx.$$

Notes:

Example 10.1.4 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 10.1.7.

Solution The setup turns out to be easier than the resulting integral. Using Key Idea 10.1.2, we have the surface area SA is:

$$\begin{aligned}
\text{SA} &= 2\pi\int_0^{\pi} \sin x\sqrt{1+\cos^2 x}\,dx \\
&= -2\pi\int_1^{-1} \sqrt{1+u^2}\,du && \text{substitute } u = \cos x \\
&= 2\pi\int_{-\pi/4}^{\pi/4} \sec^3\theta\,d\theta && \text{substitute } u = \tan\theta \\
&= \pi\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_{-\pi/4}^{\pi/4} && \text{by Example 8.2.6} \\
&= \pi\left(\sqrt{2}+\ln\left(\sqrt{2}+1\right) - \left(-\sqrt{2}+\ln\left(\sqrt{2}-1\right)\right)\right) \\
&= \pi\left(2\sqrt{2}+\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) \\
&= 2\pi\left(\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right)\text{ units}^2 && \text{rationalize the denominator.}
\end{aligned}$$

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and a natural logarithm.

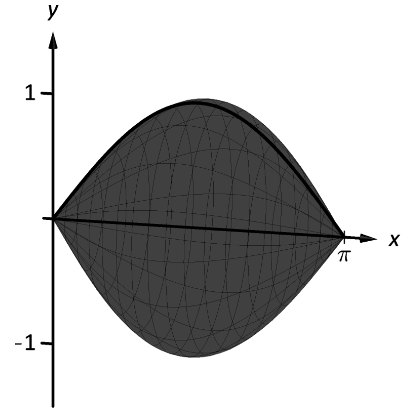


Figure 10.1.7: Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.

Example 10.1.5 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about:

1. the x-axis
2. the y-axis.

Solution

1. The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 10.1.8(a). Like the integral in Example 10.1.4, this integral is easier to setup than to actually integrate. While it is possible to use a trigonometric substitution to evaluate this integral, it is significantly more difficult than

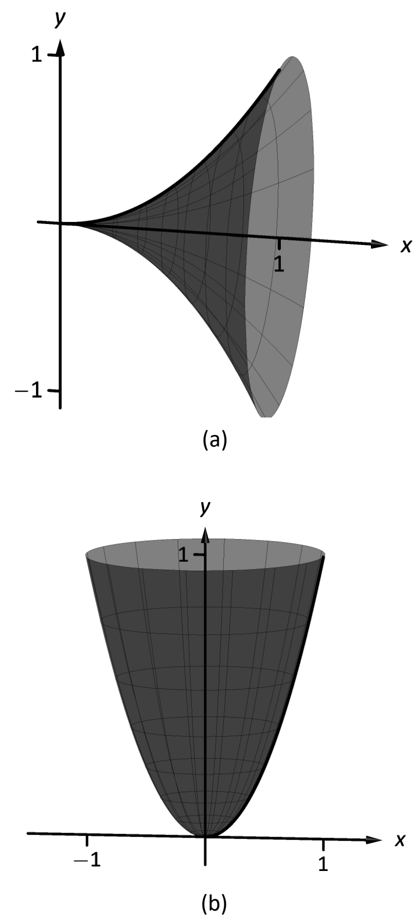


Figure 10.1.8: The solids used in Example 10.1.5.

Notes:

a solution employing the hyperbolic sine:

$$\begin{aligned} SA &= 2\pi \int_0^1 x^2 \sqrt{1+(2x)^2}\, dx. \\ &= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1} 2\right)\ \text{units}^2. \end{aligned}$$

2. Since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$\begin{aligned} SA &= 2\pi \int_0^1 x\sqrt{1+(2x)^2}\, dx \\ &= \frac{\pi}{4}\int_1^5 \sqrt{u}\, du \qquad \text{substitute } u = 1+4x^2 \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5}-1\right)\ \text{units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 10.1.8 (b).

Our final example is a famous mathematical "paradox."

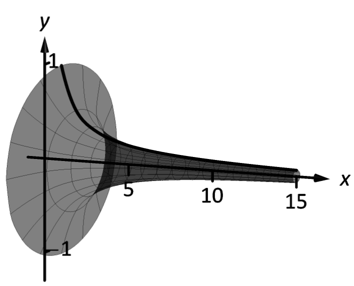


Figure 10.1.9: A graph of Gabriel's Horn.

Example 10.1.6 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 10.1.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

Solution To compute the volume it is natural to use the Disk Method.

Notes:

We have:

$$\begin{aligned} V &= \pi \int_1^{\infty} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \int_1^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \left(\frac{-1}{x}\right)\Bigg|_1^b \\ &= \lim_{b\to\infty} \pi \left(1-\frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1,\infty)$, we can state that

$$2\pi \int_1^{\infty} \frac{1}{x}\,dx < 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

By Key Idea 8.6.1, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1,\infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

Notes:

Exercises 10.1

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.
2. T/F: The integral formula for computing Arc Length includes a square-root, meaning the integration is probably easy.

Problems

In Exercises 3–12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.
4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.
5. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.
6. $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1, 4]$.
7. $f(x) = 2x^{3/2} - \frac{1}{6}\sqrt{x}$ on $[0, 9]$.
8. $f(x) = \cosh x$ on $[-\ln 2, \ln 2]$.
9. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.
10. $f(x) = \frac{1}{12}x^5 + \frac{1}{5x^3}$ on $[.1, 1]$.
11. $f(x) = \ln\big(\sin x\big)$ on $[\pi/6, \pi/2]$.
12. $f(x) = \ln\big(\cos x\big)$ on $[0, \pi/4]$.

In Exercises 13–20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.
14. $f(x) = x^{10}$ on $[0, 1]$.
15. $f(x) = \sqrt{x}$ on $[0, 1]$.
16. $f(x) = \ln x$ on $[1, e]$.
17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)
18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)
19. $f(x) = \frac{1}{x}$ on $[1, 2]$.
20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21–28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.
22. $f(x) = x^{10}$ on $[0, 1]$.
23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)
24. $f(x) = \ln x$ on $[1, e]$.
25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)
26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)
27. $f(x) = \frac{1}{x}$ on $[1, 2]$.
28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29–32, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.
30. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.
31. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.
32. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

10.2 Parametric Equations

We are familiar with sketching shapes, such as parabolas, by following this basic procedure:

$$\begin{array}{c}\text{Choose}\\ x\end{array} \longrightarrow \begin{array}{c}\text{Use a function}\\ f \text{ to find } y\\ \big(y=f(x)\big)\end{array} \longrightarrow \begin{array}{c}\text{Plot point}\\ (x,y)\end{array}$$

In the rectangular coordinate system, the **rectangular equation** $y = f(x)$ works well for some shapes like a parabola with a vertical axis of symmetry, but in Precalculus and the review of conic sections in Section 10.0, we encountered several shapes that could not be sketched in this manner. (To plot an ellipse using the above procedure, we need to plot the "top" and "bottom" separately.)

In this section we introduce a new sketching procedure:

$$\begin{tikzcd} & \begin{array}{c}\text{Use a function}\\ f \text{ to find } x\\ \big(x=f(t)\big)\end{array} \arrow[dr] & \\ \begin{array}{c}\text{Choose}\\ t\end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c}\text{Plot point}\\ (x,y)\end{array} \\ & \begin{array}{c}\text{Use a function}\\ g \text{ to find } y\\ \big(y=g(t)\big)\end{array} \arrow[ur] & \end{tikzcd}$$

Here, x and y are found separately but then plotted together. This leads us to a definition.

> **Definition 10.2.1 Parametric Equations and Curves**
> Let f and g be continuous functions on an interval I. The **graph** of the **parametric equations** $x = f(t)$ and $y = g(t)$ is the set of all points $\big(x, y\big) = \big(f(t), g(t)\big)$ in the Cartesian plane, as the **parameter** t varies over I. A **curve** is a graph along with the parametric equations that define it.

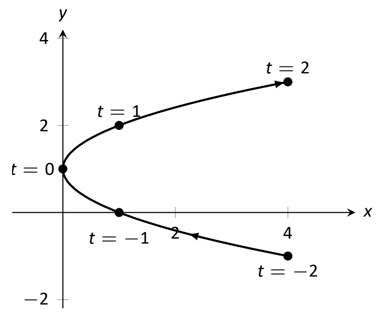
> Watch the video:
> Parametric Equations — Some basic questions at
> `https://youtu.be/9kKZHQtYP7g`

Notes:

This is a formal definition of the word *curve*. When a curve lies in a plane (such as the Cartesian plane), it is often referred to as a **plane curve**. Examples will help us understand the concepts introduced in the definition.

t	x	y
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3

(a)



(b)

Figure 10.2.1: A table of values of the parametric equations in Example 10.2.1 along with a sketch of their graph.

Example 10.2.1 Plotting parametric functions

Plot the graph of the parametric equations $x = t^2$, $y = t + 1$ for t in $[-2, 2]$.

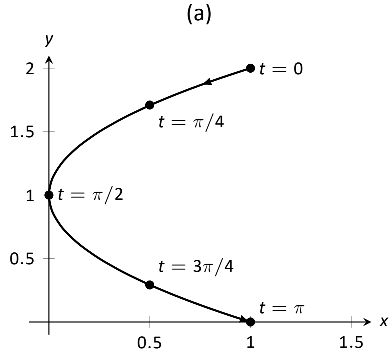
Solution We plot the graphs of parametric equations in much the same manner as we plotted graphs of functions like $y = f(x)$: we make a table of values, plot points, then connect these points with a "reasonable" looking curve. Figure 10.2.1(a) shows such a table of values; note how we have 3 columns.

The points (x, y) from the table are plotted in Figure 10.2.1(b). The points have been connected with a smooth curve. Each point has been labeled with its corresponding t-value. These values, along with the two arrows along the curve, are used to indicate the **orientation** of the graph. This information describes the **path** of a particle traveling along the curve.

We often use the letter t as the parameter as we often regard t as representing *time*. Certainly there are many contexts in which the parameter is not time, but it can be helpful to think in terms of time as one makes sense of parametric plots and their orientation (for instance, "At time $t = 0$ the position is $(1, 2)$ and at time $t = 3$ the position is $(5, 1)$.").

t	x	y
0	1	2
$\pi/4$	$1/2$	$1+\sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$1/2$	$1-\sqrt{2}/2$
π	1	0

(a)



(b)

Figure 10.2.2: A table of values of the parametric equations in Example 10.2.2 along with a sketch of their graph.

Example 10.2.2 Plotting parametric functions

Sketch the graph of the parametric equations $x = \cos^2 t$, $y = \cos t + 1$ for t in $[0, \pi]$.

Solution We again start by making a table of values in Figure 10.2.2(a), then plot the points (x, y) on the Cartesian plane in Figure 10.2.2(b).

The curves in Examples 10.2.1 and 10.2.2 are portions of the same parabola $(y-1)^2 + x = 1$. While the *parabola* is the same, the *curves* are different. In Example 10.2.1, if we let t vary over all real numbers, we'd obtain the entire parabola. In this example, letting t vary over all real numbers would still produce the same graph; this portion of the parabola would be traced, and re-traced, infinitely often. The orientation shown in Figure 10.2.2 shows the orientation on $[0, \pi]$, but this orientation is reversed on $[\pi, 2\pi]$.

Notes:

Converting between rectangular and parametric equations

It is sometimes useful to transform rectangular form equations (i.e., $y = f(x)$) into parametric form equations, and vice-versa. Converting from rectangular to parametric can be very simple: given $y = f(x)$, the parametric equations $x = t$, $y = f(t)$ produce the same graph. As an example, given $y = x^2 - x - 6$, the parametric equations $x = t, y = t^2 - t - 6$ produce the same parabola. However, other parameterizations can be used. The following example demonstrates one possible alternative.

Example 10.2.3 Converting from rectangular to parametric
Find parametric equations for $f(x) = x^2 - x - 6$.

Solution Solution 1: For any choice for x we can determine the corresponding y by substitution. If we choose $x = t-1$ then $y = (t-1)^2-(t-1)-6 = t^2 - 3t - 4$. Thus $f(x)$ can be represented by the parametric equations

$$x = t - 1 \qquad y = t^2 - 3t - 4.$$

On the graph of this parameterization (Figure 10.2.3(a)) the points have been labeled with the corresponding t-values and arrows indicate the path of a particle traveling on this curve. The particle would move from the upper left, down to the vertex at $(.5, -6.25)$ and then up to the right.

Solution 2: If we choose $x = 3 - t$ then $y = (3-t)^2 - (3-t) - 6 = t^2 - 5t$. Thus $f(x)$ can also be represented by the parametric equations

$$x = 3 - t \qquad y = t^2 - 5t.$$

On the graph of this parameterization (Figure 10.2.3(b)) the points have been labeled with the corresponding $t-$values and arrows indicate the path of a particle traveling on this curve. The particle would move down from the upper right, to the vertex at $(.5, -6.25)$ and then up to the left.

Solution 3: We can also parameterize any $y = f(x)$ by setting $t = \frac{dy}{dx}$. That is, $t = a$ corresponds to the point on the graph whose tangent line has a slope a. Computing $\frac{dy}{dx}, f'(x) = 2x-1$ we set $t = 2x-1$. Solving for x we find $x = \frac{t+1}{2}$ and by substitution $y = \frac{1}{4}t^2 - \frac{25}{4}$. Thus $f(x)$ can be represented by the parametric equations

$$x = \frac{t+1}{2} \qquad y = \frac{1}{4}t^2 - \frac{25}{4}.$$

The graph of this parameterization is shown in Figure 10.2.3(c). To find the point where the tangent line has a slope of 0, we set $t = 0$. This gives us the point $(.5, -6.25)$ which is the vertex of $f(x)$.

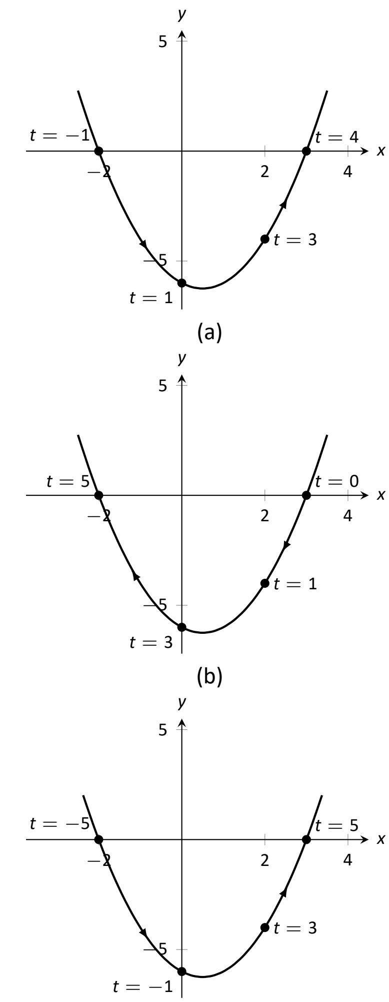


Figure 10.2.3: The equation $f(x) = x^2 - x - 6$ with different parameterizations.

Notes:

Example 10.2.4 Converting from rectangular to parametric

Find parametric equations for the circle $x^2 + y^2 = 4$.

SOLUTION We will present three different approaches:

Solution 1: Consider the equivalent equation $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ and the Pythagorean Identity, $\sin^2 t + \cos^2 t = 1$. We set $\cos t = \frac{x}{2}$ and $\sin t = \frac{y}{2}$, which gives $x = 2\cos t$ and $y = 2\sin t$. To trace the circle once, we must have $0 \le t \le 2\pi$. Note that when $t = 0$ a particle tracing the curve would be at the point $(2, 0)$ and would move in a counterclockwise direction.

Solution 2: Another parameterization of the same circle would be $x = 2\sin t$ and $y = 2\cos t$ for $0 \le t \le 2\pi$. When $t = 0$ a particle would be at the point $(0, 2)$ and would move in a clockwise direction.

Solution 3: We could let $x = -2\sin t$ and $y = 2\cos t$ for $0 \le t \le 2\pi$. Also note that we could use $x = 2\cos 2t$ and $y = 2\sin 2t$ for $0 \le t \le \pi$.

As we have shown in the previous examples, there are many different ways to parameterize any given curve. We sometimes choose the parameter to accurately model physical behavior.

Example 10.2.5 Converting from rectangular to parametric

Find a parameterization that traces the ellipse $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1$ starting at the point $(-1, -3)$ in a clockwise direction.

SOLUTION Applying the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, we set $\cos^2 t = \frac{(x-2)^2}{9}$ and $\sin^2 t = \frac{(y+3)^2}{4}$. Solving these equations for x and y we set $x = -3\cos t + 2$ and $y = 2\sin t - 3$ for $0 \le t \le 2\pi$.

Example 10.2.6 Converting from rectangular to parametric

Find a parameterization for the hyperbola $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 1$.

SOLUTION We use the form of the Pythagorean Identity $\sec^2 t - \tan^2 t = 1$. We let $\sec^2 t = \frac{(x-2)^2}{9}$ and $\tan^2 t = \frac{(y-3)^2}{4}$. Solving these equations for x and y we have $x = 3\sec t + 2$ and $y = 2\tan t + 3$ for $0 \le t \le 2\pi$ and $t \ne \frac{\pi}{2}, \frac{3\pi}{2}$.

Notes:

Example 10.2.7 Converting from rectangular to parametric

An object is fired from a height of 0ft and lands 6 seconds later, 192ft away. Assuming ideal projectile motion, the height, in feet, of the object can be described by $h(x) = -x^2/64 + 3x$, where x is the distance in feet from the initial location. (Thus $h(0) = h(192) = 0$ft.) Find parametric equations $x = f(t)$, $y = g(t)$ for the path of the projectile where x is the horizontal distance the object has traveled at time t (in seconds) and y is the height at time t.

SOLUTION Physics tells us that the horizontal motion of the projectile is linear; that is, the horizontal speed of the projectile is constant. Since the object travels 192ft in 6s, we deduce that the object is moving horizontally at a rate of 32ft/s, giving the equation $x = 32t$. As $y = -x^2/64 + 3x$, we find $y = -16t^2 + 96t$. We can quickly verify that $y'' = -32\text{ft/s}^2$, the acceleration due to gravity, and that the projectile reaches its maximum at $t = 3$, halfway along its path.

These parametric equations make certain determinations about the object's location easy: 2 seconds into the flight the object is at the point $\big(x(2), y(2)\big) = \big(64, 128\big)$. That is, it has traveled horizontally 64ft and is at a height of 128ft, as shown in Figure 10.2.4.

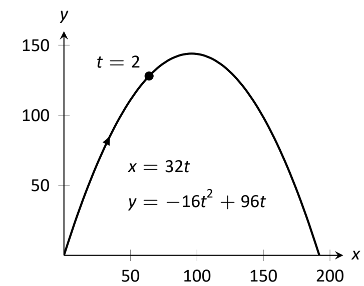


Figure 10.2.4: Graphing projectile motion in Example 10.2.7.

It is sometimes necessary to convert given parametric equations into rectangular form. This can be decidedly more difficult, as some "simple" looking parametric equations can have very "complicated" rectangular equations. This conversion is often referred to as "eliminating the parameter," as we are looking for a relationship between x and y that does not involve the parameter t.

Example 10.2.8 Eliminating the parameter

Find a rectangular equation for the curve described by

$$x = \frac{1}{t^2+1} \quad \text{and} \quad y = \frac{t^2}{t^2+1}.$$

SOLUTION There is not a set way to eliminate a parameter. One method is to solve for t in one equation and then substitute that value in the second. We use that technique here, then show a second, simpler method.

Starting with $x = 1/(t^2+1)$, solve for t: $t = \pm\sqrt{1/x - 1}$. Substitute this value for t in the equation for y:

$$\begin{aligned} y &= \frac{t^2}{t^2+1} \\ &= \frac{1/x - 1}{1/x - 1 + 1} \end{aligned}$$

Notes:

$$= \frac{1/x - 1}{1/x}$$
$$= \left(\frac{1}{x} - 1\right) \cdot x$$
$$= 1 - x.$$

Thus $y = 1 - x$. One may have recognized this earlier by manipulating the equation for y:

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1 - x.$$

This is a shortcut that is very specific to this problem; sometimes shortcuts exist and are worth looking for.

We should be careful to limit the domain of the function $y = 1 - x$. The parametric equations limit x to values in $(0, 1]$, thus to produce the same graph we should limit the domain of $y = 1 - x$ to the same.

The graphs of these functions are given in Figure 10.2.5. The portion of the graph defined by the parametric equations is given in a thick line; the graph defined by $y = 1 - x$ with unrestricted domain is given in a thin line.

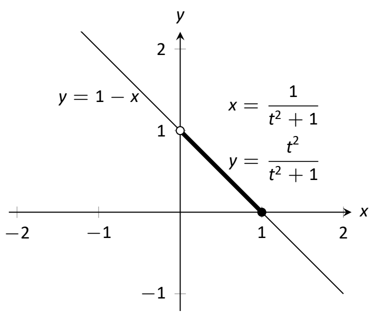


Figure 10.2.5: Graphing parametric and rectangular equations for a graph in Example 10.2.8.

Example 10.2.9 Eliminating the parameter

Eliminate the parameter in $x = 4\cos t + 3$, $y = 2\sin t + 1$

Solution We should not try to solve for t in this situation as the resulting algebra/trig would be messy. Rather, we solve for $\cos t$ and $\sin t$ in each equation, respectively. This gives

$$\cos t = \frac{x-3}{4} \quad \text{and} \quad \sin t = \frac{y-1}{2}.$$

The Pythagorean Theorem gives $\cos^2 t + \sin^2 t = 1$, so:

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$
$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$$

This final equation should look familiar — it is the equation of an ellipse. Figure 10.2.6 plots the parametric equations, demonstrating that the graph is indeed of an ellipse with a horizontal major axis and center at $(3, 1)$.

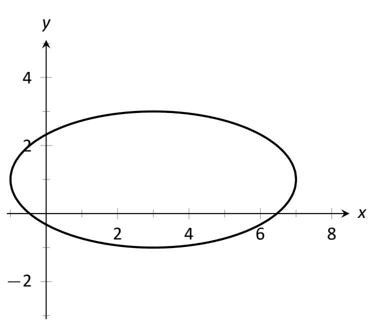


Figure 10.2.6: Graphing the parametric equations $x = 4\cos t + 3$, $y = 2\sin t + 1$ in Example 10.2.9.

Notes:

Graphs of Parametric Equations

These examples begin to illustrate the powerful nature of parametric equations. Their graphs are far more diverse than the graphs of functions produced by "$y = f(x)$" functions.

One nice feature of parametric equations is that their graphs are easy to shift. While this is not too difficult in the "$y = f(x)$" context, the resulting function can look rather messy. (Plus, to shift to the right by two, we replace x with $x - 2$, which is counterintuitive.) The following example demonstrates this.

Example 10.2.10 Shifting the graph of parametric functions

Sketch the graph of the parametric equations $x = t^2 + t$, $y = t^2 - t$. Find new parametric equations that shift this graph to the right 3 units and down 2.

Solution We see the graph in Figure 10.2.7(a). It is a parabola with an axis of symmetry along the line $y = x$; the vertex is at $(0, 0)$. It should be noted that finding the vertex is not a trivial matter and not something you will be asked to do in this text.

In order to shift the graph to the right 3 units, we need to increase the x-value by 3 for every point. The straightforward way to accomplish this is simply to add 3 to the function defining x: $x = t^2 + t + 3$. To shift the graph down by 2 units, we wish to decrease each y-value by 2, so we subtract 2 from the function defining y: $y = t^2 - t - 2$. Thus our parametric equations for the shifted graph are $x = t^2 + t + 3$, $y = t^2 - t - 2$. This is graphed in Figure 10.2.7 (b). Notice how the vertex is now at $(3, -2)$.

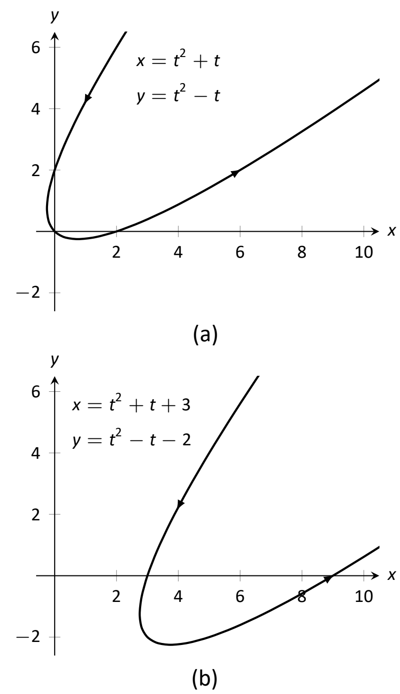


Figure 10.2.7: Illustrating how to shift graphs in Example 10.2.10.

Because the x- and y-values of a graph are determined independently, the graphs of parametric functions often possess features not seen on "$y = f(x)$" type graphs. The next example demonstrates how such graphs can arrive at the same point more than once.

Example 10.2.11 Graphs that cross themselves

Plot the parametric functions $x = t^3 - 5t^2 + 3t + 11$ and $y = t^2 - 2t + 3$ and determine the t-values where the graph crosses itself.

Solution Using the methods developed in this section, we again plot points and graph the parametric equations as shown in Figure 10.2.8. It appears that the graph crosses itself at the point $(2, 6)$, but we'll need to analytically determine this.

We are looking for two different values, say, s and t, where $x(s) = x(t)$ and $y(s) = y(t)$. That is, the x-values are the same precisely when the y-values are

Notes:

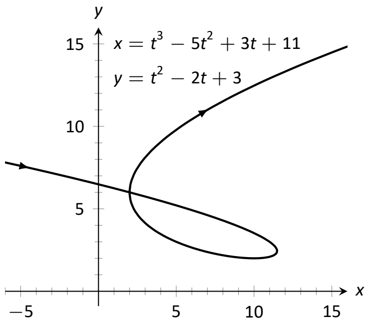


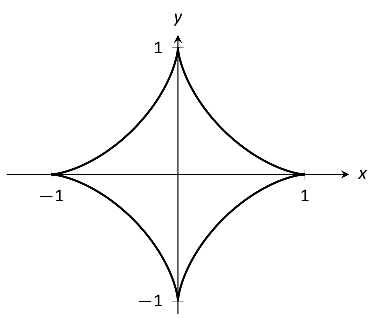
Figure 10.2.8: A graph of the parametric equations from Example 10.2.11.

the same. This gives us a system of 2 equations with 2 unknowns:

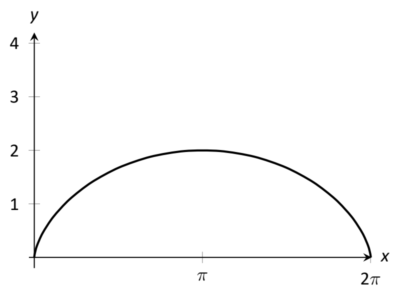
$$s^3 - 5s^2 + 3s + 11 = t^3 - 5t^2 + 3t + 11$$
$$s^2 - 2s + 3 = t^2 - 2t + 3$$

Solving this system is not trivial but involves only algebra. Using the quadratic formula, one can solve for t in the second equation and find that $t = 1 \pm \sqrt{s^2 - 2s + 1}$. This can be substituted into the first equation, revealing that the graph crosses itself at $t = -1$ and $t = 3$. We confirm our result by computing $x(-1) = x(3) = 2$ and $y(-1) = y(3) = 6$.

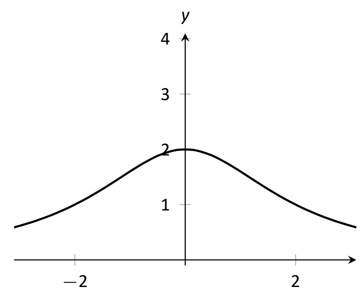
We now present a small gallery of "interesting" and "famous" curves along with parametric equations that produce them.



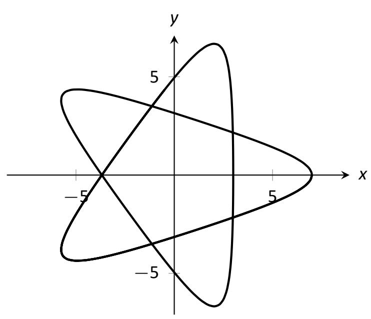
Astroid
$x = \cos^3 t$
$y = \sin^3 t$



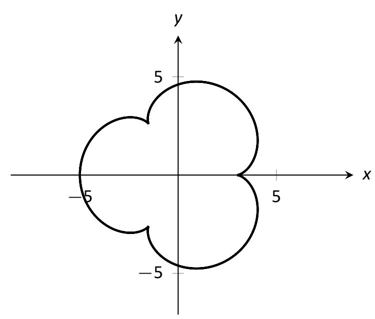
Cycloid
$x = r(t - \sin t)$
$y = r(1 - \cos t)$



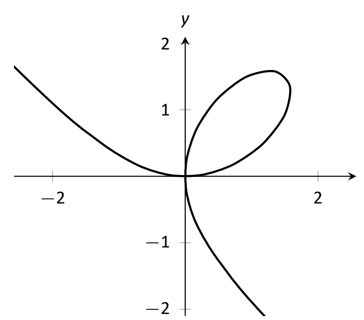
Witch of Agnesi
$x = 2at$
$y = 2a/(1 + t^2)$



Hypotrochoid
$x = 2\cos(t) + 5\cos(2t/3)$
$y = 2\sin(t) - 5\sin(2t/3)$



Epicycloid
$x = 4\cos(t) - \cos(4t)$
$y = 4\sin(t) - \sin(4t)$



Folium of Descartes
$x = 3at/(1 + t^3)$
$y = 3at^2/(1 + t^3)$

Notes:

One might note a feature shared by three of these graphs: "sharp corners," or **cusps**. We have seen graphs with cusps before and determined that such functions are not differentiable at these points. This leads us to a definition.

Definition 10.2.2 Smooth
A curve C defined by $x = f(t)$, $y = g(t)$ is **smooth** on an interval I if f' and g' are continuous on I and not simultaneously 0 (except possibly at the endpoints of I). A curve is **piecewise smooth** on I if I can be partitioned into subintervals where C is smooth on each subinterval.

Consider the astroid, given by $x = \cos^3 t$, $y = \sin^3 t$. Taking derivatives, we have:

$$x' = -3\cos^2 t \sin t \quad \text{and} \quad y' = 3\sin^2 t \cos t.$$

It is clear that each is 0 when $t = 0,\ \pi/2,\ \pi, \ldots$. Thus the astroid is not smooth at these points, corresponding to the cusps seen in the figure. However, by restricting the domain of the astroid to all reals except $t = \frac{k\pi}{2}$ for $k \in \mathbb{Z}$ we have a piecewise smooth curve.

We demonstrate this once more.

Example 10.2.12 Determine where a curve is not smooth
Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

Solution We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

We set each equal to 0:

$$x' = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2$$
$$y' = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$$

We consider only the value of $t = 2$ since both x' and y' must be 0. Thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 10.2.9, illustrating the cusp at $(1, 4)$.

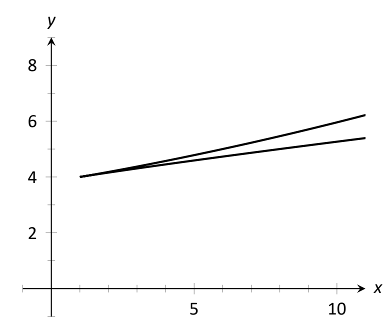


Figure 10.2.9: Graphing the curve in Example 10.2.12; note it is not smooth at $(1, 4)$.

If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change of x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some object, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving objects tend to change direction in a "smooth" fashion.

Notes:

Example 10.2.13 The Cycloid

A well-known parametric curve is the cycloid. Fix r, and let $x = r(t - \sin t)$, $y = r(1 - \cos t)$. This represents the path traced out by a point on a wheel of radius r as it starts rolling to the right. We can think of t as the angle through which the point has rotated.

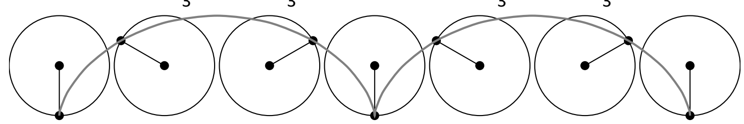


Figure 10.2.10: A cycloid traced through two revolutions.

Figure 10.2.10 shows a cycloid sketched out with the wheel shown at various places. The dot on the rim is the point on the wheel that we're using to trace out the curve.

From this sketch we can see that one arch of the cycloid is traced out in the range $0 \leq t \leq 2\pi$. This makes sense when you consider that the point will be back on the ground after it has rotated through an angle of 2π.

One should be careful to note that a "sharp corner" does not have to occur when a curve is not smooth. For instance, one can verify that $x = t^3$, $y = t^6$ produce the familiar $y = x^2$ parabola. However, in this parameterization, the curve is not smooth. A particle traveling along the parabola according to the given parametric equations comes to rest at $t = 0$, though no sharp point is created.

Our previous experience with cusps taught us that a function was not differentiable at a cusp. This can lead us to wonder about derivatives in the context of parametric equations and the application of other calculus concepts. Given a curve defined parametrically, how do we find the slopes of tangent lines? Can we determine concavity? We explore these concepts and more in the next section.

Notes:

Exercises 10.2

Terms and Concepts

1. T/F: When sketching the graph of parametric equations, the x and y values are found separately, then plotted together.
2. The direction in which a graph is "moving" is called the ____ of the graph.
3. An equation written as $y = f(x)$ is written in ____ form.
4. Create parametric equations $x = f(t)$, $y = g(t)$ and sketch their graph. Explain any interesting features of your graph based on the functions f and g.

Problems

In Exercises 5–8, sketch the graph of the given parametric equations **by hand**, making a table of points to plot. Be sure to indicate the orientation of the graph.

5. $x = t^2 + t, \quad y = 1 - t^2, \quad -3 \leq t \leq 3$
6. $x = 1, \quad y = 5\sin t, \quad -\pi/2 \leq t \leq \pi/2$
7. $x = t^2, \quad y = 2, \quad -2 \leq t \leq 2$
8. $x = t^3 - t + 3, \quad y = t^2 + 1, \quad -2 \leq t \leq 2$

In Exercises 9–18, sketch the graph of the given parametric equations; using a graphing utility is advisable. Be sure to indicate the orientation of the graph.

9. $x = t^3 - 2t^2, \quad y = t^2, \quad -2 \leq t \leq 3$
10. $x = 1/t, \quad y = \sin t, \quad 0 < t \leq 10$
11. $x = 3\cos t, \quad y = 5\sin t, \quad 0 \leq t \leq 2\pi$
12. $x = 3\cos t + 2, \quad y = 5\sin t + 3, \quad 0 \leq t \leq 2\pi$
13. $x = \cos t, \quad y = \cos(2t), \quad 0 \leq t \leq \pi$
14. $x = \cos t, \quad y = \sin(2t), \quad 0 \leq t \leq 2\pi$
15. $x = 2\sec t, \quad y = 3\tan t, \quad -\pi/2 < t < \pi/2$
16. $x = \cosh t, \quad y = \sinh t, \quad -2 \leq t \leq 2$
17. $x = \cos t + \frac{1}{4}\cos(8t), \quad y = \sin t + \frac{1}{4}\sin(8t), \quad 0 \leq t \leq 2\pi$
18. $x = \cos t + \frac{1}{4}\sin(8t), \quad y = \sin t + \frac{1}{4}\cos(8t), \quad 0 \leq t \leq 2\pi$

In Exercises 19–20, four sets of parametric equations are given. Describe how their graphs are similar and different. Be sure to discuss orientation and ranges.

19. (a) $x = t \quad y = t^2, \quad -\infty < t < \infty$
    (b) $x = \sin t \quad y = \sin^2 t, \quad -\infty < t < \infty$
    (c) $x = e^t \quad y = e^{2t}, \quad -\infty < t < \infty$
    (d) $x = -t \quad y = t^2, \quad -\infty < t < \infty$
20. (a) $x = \cos t \quad y = \sin t, \quad 0 \leq t \leq 2\pi$
    (b) $x = \cos(t^2) \quad y = \sin(t^2), \quad 0 \leq t \leq 2\pi$
    (c) $x = \cos(1/t) \quad y = \sin(1/t), \quad 0 < t < 1$
    (d) $x = \cos(\cos t) \quad y = \sin(\cos t), \quad 0 \leq t \leq 2\pi$

In Exercises 21–24, find a parameterization for the curve.

21. $y = 9 - 4x$
22. $4x - y^2 = 5$
23. $(x + 9)^2 + (y - 4)^2 = 49$
24. $(x - 2)^2 - (y + 3)^2 = 25$

In Exercises 25–28, find parametric equations and a parameter interval.

25. The line segment with endpoints $(-1, -3)$ and $(4, 1)$
26. The line segment with endpoints $(-1, 3)$ and $(3, -2)$
27. The left half of the parabola $y = x^2 + 2x$
28. The lower half of the parabola $x = 1 - y^2$

In Exercises 29–32, find parametric equations for the given rectangular equation using the parameter $t = \dfrac{dy}{dx}$. Verify that at $t = 1$, the point on the graph has a tangent line with slope of 1.

29. $y = 3x^2 - 11x + 2$
30. $y = e^x$
31. $y = \sin x$ on $[0, \pi]$
32. $y = \sqrt{x}$ on $[0, \infty)$
33. Find parametric equations and a parameter interval for the motion of a particle that starts at $(1, 0)$ and traces the circle $x^2 + y^2 = 1$
    (a) once clockwise
    (b) once counter-clockwise
    (a) twice clockwise
    (b) twice counter-clockwise
34. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
    (a) once clockwise
    (b) once counter-clockwise
    (a) twice clockwise
    (b) twice counter-clockwise

In Exercises 35–44, find parametric equations that describe the given situation.

35. A projectile is fired from a height of 0 ft, landing 16 ft away in 4 s.
36. A projectile is fired from a height of 0 ft, landing 200 ft away in 4 s.
37. A projectile is fired from a height of 0 ft, landing 200 ft away in 20 s.
38. A circle of radius 2, centered at the origin, that is traced clockwise once on $[0, 2\pi]$.
39. A circle of radius 3, centered at $(1, 1)$, that is traced once counter-clockwise on $[0, 1]$.
40. An ellipse centered at $(1, 3)$ with vertical major axis of length 6 and minor axis of length 2.
41. An ellipse with foci at $(\pm 1, 0)$ and vertices at $(\pm 5, 0)$.
42. A hyperbola with foci at $(5, -3)$ and $(-1, -3)$, and with vertices at $(1, -3)$ and $(3, -3)$.
43. A hyperbola with vertices at $(0, \pm 6)$ and asymptotes $y = \pm 3x$.
44. A lug nut that is 2" from the center of a car tire. The tire is 18" in diameter and rolling at a speed of 10"/sec.

In Exercises 45–54, eliminate the parameter in the given parametric equations.

45. $x = 2t + 5, \quad y = -3t + 1$
46. $x = \sec t, \quad y = \tan t$
47. $x = 4\sin t + 1, \quad y = 3\cos t - 2$
48. $x = t^2, \quad y = t^3$
49. $x = \dfrac{1}{t+1}, \quad y = \dfrac{3t+5}{t+1}$
50. $x = e^t, \quad y = e^{3t} - 3$
51. $x = \ln t, \quad y = t^2 - 1$
52. $x = \cot t, \quad y = \csc t$
53. $x = \cosh t, \quad y = \sinh t$
54. $x = \cos(2t), \quad y = \sin t$

In Exercises 55–58, eliminate the parameter in the given parametric equations. Describe the curve defined by the parametric equations based on its rectangular form.

55. $x = at + x_0, \quad y = bt + y_0$
56. $x = r\cos t, \quad y = r\sin t$
57. $x = a\cos t + h, \quad y = b\sin t + k$
58. $x = a\sec t + h, \quad y = b\tan t + k$

In Exercises 59–62, find the values of t where the graph of the parametric equations crosses itself.

59. $x = t^3 - t + 3, \quad y = t^2 - 3$
60. $x = t^3 - 4t^2 + t + 7, \quad y = t^2 - t$
61. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$
62. $x = \cos t\cos(3t), \quad y = \sin t\cos(3t)$ on $[0, \pi]$

In Exercises 63–66, find the value(s) of t where the curve defined by the parametric equations is not smooth.

63. $x = t^3 + t^2 - t, \quad y = t^2 + 2t + 3$
64. $x = t^2 - 4t, \quad y = t^3 - 2t^2 - 4t$
65. $x = \cos t, \quad y = 2\cos t$
66. $x = 2\cos t - \cos(2t), \quad y = 2\sin t - \sin(2t)$

10.3 Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

Key Idea 10.3.1 Finding $\frac{dy}{dx}$ with Parametric Equations.
Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 10.3.1 Tangent Lines
Let a curve C be parameterized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line

$$f'(t_0)(y - g(t_0)) = g'(t_0)(x - f(t_0)).$$

Notice that the tangent line goes through the point $(f(t_0), g(t_0))$. It is possible for parametric curves to have horizontal and vertical tangents. As expected a horizontal tangent occurs whenever $\frac{dy}{dx} = 0$ or when $\frac{dy}{dt} = 0$ (provided $\frac{dx}{dt} \neq 0$).

Notes:

Similarly, a vertical tangent occurs whenever $\frac{dy}{dx}$ is undefined or when $\frac{dx}{dt} = 0$ (provided $\frac{dy}{dt} \neq 0$). When $\frac{dy}{dx}$ is defined, however, the tangent line has slope $\frac{g'(t_0)}{f'(t_0)}$.

Definition 10.3.2 Normal Lines

The **normal line** to a curve C at a point P is the line through P and perpendicular to the tangent line at P. This has equation

$$g'(t_0)(y - g(t_0)) = -f'(t_0)(x - f(t_0)).$$

As with the tangent line we note that it is possible for a normal line to be vertical or horizontal. A horizontal normal line occurs whenever $\frac{dy}{dx}$ is undefined or when $\frac{dx}{dt} = 0$ (provided $\frac{dy}{dt} \neq 0$). Similarly, a vertical normal line occurs whenever $\frac{dy}{dx} = 0$ or when $\frac{dy}{dt} = 0$ (provided $\frac{dx}{dt} \neq 0$). In other words, if the curve C has a vertical tangent at $(f(t_0), g(t_0))$ the normal line will be horizontal and if the tangent is horizontal the normal line will be a vertical line.

Watch the video:
Derivatives of Parametric Functions at
`https://youtu.be/k5QnaGVk1JI`

Example 10.3.1 Tangent and Normal Lines to Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.
2. Find where C has vertical and horizontal tangent lines.

Solution

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t + 6}{10t - 6}.$$

Notes:

Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines.

The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

- the equation of the tangent line is $y = \dfrac{1}{2}(x - 31) + 26$, and
- the equation of the normal line is $y = -2(x - 31) + 26$.

This is illustrated in Figure 10.3.1.

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \quad \Rightarrow \quad 2t + 6 = 0 \quad \Rightarrow \quad t = -3.$$

The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$).

To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \quad \Rightarrow \quad 10t - 6 = 0 \quad \Rightarrow \quad t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$.

The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 10.3.1.

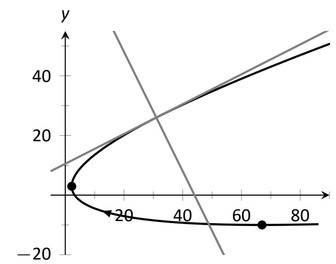


Figure 10.3.1: Graphing tangent and normal lines in Example 10.3.1.

Example 10.3.2 Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by $x = \cos t$ and $y = \sin t$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.
2. Find the equation of the normal line at $t = t_0$.

SOLUTION

Notes:

1. We compute the derivative following Key Idea 10.3.1:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos t}{\sin t}.$$

The derivative is 0 when $\cos t = 0$; that is, when $t = \pi/2,\ 3\pi/2$. These are the points $(0,1)$ and $(0,-1)$ on the circle.

The normal line is horizontal (and hence, the tangent line is vertical) when $\sin t = 0$; that is, when $t = 0,\ \pi,\ 2\pi$, corresponding to the points $(-1,0)$ and $(0,1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t = t_0$ is $m = \dfrac{\sin t_0}{\cos t_0} = \tan t_0$. This normal line goes through the point $(\cos t_0, \sin t_0)$, giving the line

$$\begin{aligned} y &= \frac{\sin t_0}{\cos t_0}(x - \cos t_0) + \sin t_0 \\ &= (\tan t_0)x, \end{aligned}$$

as long as $\cos t_0 \neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 10.3.2. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.

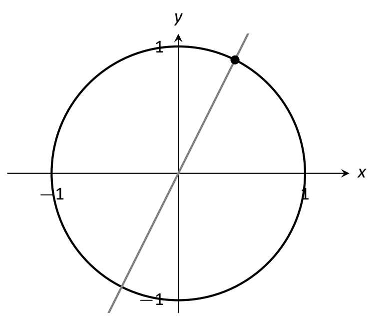


Figure 10.3.2: Illustrating how a circle's normal lines pass through its center.

Example 10.3.3 Tangent lines when $\frac{dy}{dx}$ is not defined

Find the equation of the tangent line to the astroid $x = \cos^3 t$, $y = \sin^3 t$ at $t = 0$, shown in Figure 10.3.3.

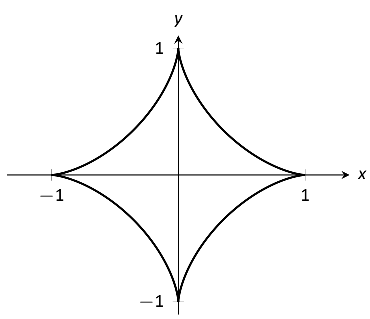


Figure 10.3.3: A graph of an astroid.

Solution We start by finding $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin t\cos^2 t, \qquad y'(t) = 3\cos t\sin^2 t.$$

Note that both of these are 0 at $t = 0$; the curve is not smooth at $t = 0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near $t = 0$, and take the limit as $t \to 0$.

$$\begin{aligned} \lim_{t\to 0}\frac{y'(t)}{x'(t)} &= \lim_{t\to 0}\frac{3\cos t\sin^2 t}{-3\sin t\cos^2 t} \quad \text{(We can reduce as } t \neq 0.) \\ &= \lim_{t\to 0}\left(-\frac{\sin t}{\cos t}\right) \\ &= 0. \end{aligned}$$

Notes:

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t = t_0$, we can define its value by setting it to be $\lim_{t \to t_0} \frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.

We found the slope of the tangent line at $t = 0$ to be 0; therefore the tangent line is $y = 0$, the x-axis.

Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx} \cdot \frac{dx}{dt} \quad \Rightarrow \quad \frac{dh}{dx} = \frac{dh/\,dt}{dx/\,dt}.$$

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to t*, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 10.3.2 Finding $\frac{d^2y}{dx^2}$ with Parametric Equations

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{f'(t)}.$$

Examples will help us understand this Key Idea.

Notes:

Example 10.3.4 Concavity of Plane Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 10.3.1. Determine the t-intervals on which the graph is concave up/down.

Solution Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 10.3.2.

In Example 10.3.1, we found $\frac{dy}{dx} = \frac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left[\frac{2t+6}{10t-6}\right]}{10t-6} \\ &= \frac{-\frac{72}{(10t-6)^2}}{10t-6} \\ &= -\frac{72}{(10t-6)^3} \\ &= -\frac{9}{(5t-3)^3}\end{aligned}$$

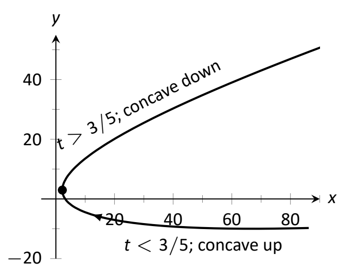


Figure 10.3.4: Graphing the parametric equations in Example 10.3.4 to demonstrate concavity.

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\frac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t - 3 = 0$; that is, when $t = 3/5$. Following the work established in Section 3.4, we look at values of t greater or less than $3/5$ on a number line:

x	$< \frac{3}{5}$	$> \frac{3}{5}$
f''	$+$	$-$
f	CU	CD

Reviewing Example 10.3.1, we see that when $t = 3/5 = 0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 10.3.4.

Notes:

Example 10.3.5 Concavity of Plane Curves

Find the points of inflection of the graph of the parametric equations $x = \sqrt{t}$, $y = \sin t$, for $0 \le t \le 16$.

SOLUTION We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t} - 2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t - 4t\sin t.$$

The possible points of inflection are found by setting $\frac{d^2y}{dx^2} = 0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 10.3.5(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t = 0.5,\ 3.5,\ 6.5,\ 9.5,\ 12.5$ and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 10.3.5(b). Note how most occur near the x-axis, but not exactly on the axis.

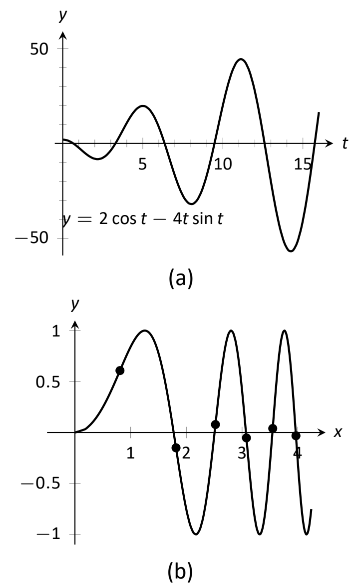


Figure 10.3.5: In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 10.3.5 along with the points of inflection.

Area with Parametric Equations

We will now find a formula for determining the area under a parametric curve given by the parametric equations

$$x = f(t) \qquad y = g(t).$$

We will also need to further add in the assumption that the curve is traced out exactly once as t increases from α to β. First, recall how to find the area under $y = F(x)$ on $a \le x \le b$:

$$A = \int_a^b F(x)\,dx.$$

Now think of the parametric equation $x = f(t)$ as a substitution in the integral, assuming that $a = f(\alpha)$ and $b = f(\beta)$ for the purposes of this formula. (There is actually no reason to assume that this will always be the case and so we'll give a corresponding formula later if it's the opposite case ($b = f(\alpha)$ and $a = f(\beta)$).)

Notes:

In order to substitute, we'll need $\mathrm{d}x = f'(t)\,\mathrm{d}t$. Plugging this into the area formula above and making sure to change the limits to their corresponding t values gives us

$$A = \int_{\alpha}^{\beta} F(f(t))f'(t)\,\mathrm{d}t.$$

Since we don't know what $F(x)$ is, we'll use the fact that

$$y = F(x) = F(f(t)) = g(t)$$

and arrive at the formula that we want.

Key Idea 10.3.3 Area Under a Parametric Curve

The area under the parametric curve given by $x = f(t)$, $y = g(t)$, for $f(\alpha) = a < x < b = f(\beta)$ is

$$A = \int_{\alpha}^{\beta} g(t)f'(t)\,\mathrm{d}t.$$

On the other hand, if we should happen to have $b = f(\alpha)$ and $a = f(\beta)$, then the formula would be

$$A = \int_{\beta}^{\alpha} g(t)f'(t)\,\mathrm{d}t.$$

Let's work an example.

Example 10.3.6 Finding the area under a parametric curve

Determine the area under the cycloid given by the parametric equations

$$x = 6(\theta - \sin\theta) \qquad y = 6(1 - \cos\theta) \qquad 0 \le \theta \le 2\pi.$$

Solution First, notice that we've switched the parameter to θ for this problem. This is to make sure that we don't get too locked into always having t as the parameter.

Now, we could graph this to verify that the curve is traced out exactly once for the given range if we wanted to.

There really isn't too much to this example other than plugging the parametric equations into the formula. We'll first need the derivative of the parametric equation for x however.

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 6(1 - \cos\theta).$$

Notes:

The area is then

$$\begin{aligned} A &= \int_0^{2\pi} 36(1-\cos\theta)^2\,d\theta \\ &= 36\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta \\ &= 36\int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta)\,d\theta \\ &= 36\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\ &= 108\pi. \end{aligned}$$

Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 10.1 we found the arc length of the graph of a function, from $x = a$ to $x = b$, to be

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

We can use this equation and convert it to the parametric equation context. Letting $x = f(t)$ and $y = g(t)$, we know that $\frac{dy}{dx} = g'(t)/f'(t)$. Suppose that $f'(t) > 0$, and calculate the differential of x:

$$dx = f'(t)\,dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)}\cdot dx.$$

Starting with the arc length formula above, consider:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_a^b \sqrt{1 + \frac{[g'(t)]^2}{[f'(t)]^2}}\,dx \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\cdot \underbrace{\frac{1}{f'(t)}\,dx}_{=dt} && \text{Factor out the } [f'(t)]^2 \\ &= \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt. \end{aligned}$$

Notes:

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a = f(t_1)$ and $b = f(t_2)$. This formula holds even when f' isn't positive and we restate it as a theorem.

Theorem 10.3.1 Arc Length of Parametric Curves
Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on some open interval I containing t_1 and t_2 on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Example 10.3.7 Arc Length of a Circle
Find the arc length of the circle parameterized by $x = 3\cos t$, $y = 3\sin t$ on $[0, 3\pi/2]$.

Solution By direct application of Theorem 10.3.1, we have

$$L = \int_0^{3\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt.$$

Apply the Pythagorean Theorem.

$$= \int_0^{3\pi/2} 3\, dt$$
$$= 3t\Big|_0^{3\pi/2} = 9\pi/2.$$

This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4 \cdot 6\pi = 9\pi/2$.

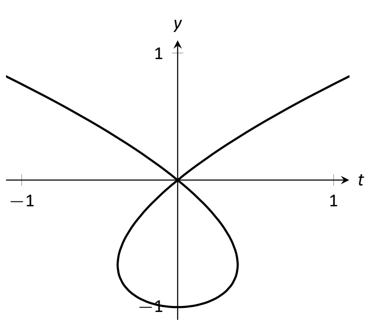


Figure 10.3.6: A graph of the parametric equations in Example 10.3.8, where the arc length of the teardrop is calculated.

Notes:

Example 10.3.8 Arc Length of a Parametric Curve
The graph of the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ crosses itself as shown in Figure 10.3.6, forming a "teardrop." Find the arc length of the teardrop.

Solution We can see by the parameterizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 10.3.1, we have

$$\begin{aligned} L &= \int_{-1}^{1} \sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt. \end{aligned}$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.

Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Key Idea 10.1.2 from Section 10.1 in a similar way as done to produce the formula for arc length done before.

Key Idea 10.3.4 Surface Area of a Solid of Revolution
Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t) \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Notes:

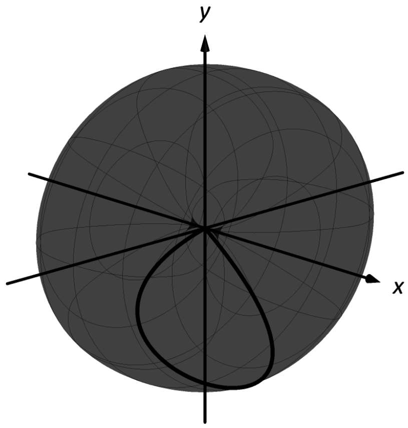


Figure 10.3.7: Rotating a teardrop shape about the x-axis in Example 10.3.9.

Example 10.3.9 Surface Area of a Solid of Revolution

Consider the teardrop shape formed by the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ as seen in Example 10.3.8. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 10.3.7.

Solution The teardrop shape is formed between $t = -1$ and $t = 1$. Using Key Idea 10.3.4, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\begin{aligned} \text{Area } S &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt. \end{aligned}$$

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equations defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.

Notes:

Exercises 10.3

Terms and Concepts

1. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$.
2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 10.3.1 is a function of ________?
3. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.
4. T/F: If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line.

Problems

In Exercises 5–12, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t, y = t^2; \quad t = 1$
6. $x = \sqrt{t}, y = 5t + 2; \quad t = 4$
7. $x = t^2 - t, y = t^2 + t; \quad t = 1$
8. $x = t^2 - 1, y = t^3 - t; \quad t = 0$ and $t = 1$
9. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$; $\quad t = \pi/4$
10. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$; $\quad t = \pi/4$
11. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$; $\quad t = 3\pi/4$
12. $x = e^{t/10} \cos t, y = e^{t/10} \sin t; \quad t = \pi/2$

In Exercises 13–20, find t-values where the curve defined by the given parametric equations has horizontal or vertical tangent lines. Note: these are the same equations as in Exercises 5–12.

13. $x = t, y = t^2$
14. $x = \sqrt{t}, y = 5t + 2$
15. $x = t^2 - t, y = t^2 + t$
16. $x = t^2 - 1, y = t^3 - t$
17. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$
18. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$
19. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$
20. $x = e^{t/10} \cos t, y = e^{t/10} \sin t$ on $[-\pi, \pi]$

In Exercises 21–24, find $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\displaystyle\lim_{t \to t_0} \frac{dy}{dx}$.

21. $x = \dfrac{1}{t^2 + 1}, \quad y = t^3$
22. $x = -t^3 + 7t^2 - 16t + 13, \quad y = t^3 - 5t^2 + 8t - 2$
23. $x = t^3 - 3t^2 + 3t - 1, \quad y = t^2 - 2t + 1$
24. $x = \cos^2 t, \quad y = 1 - \sin^2 t$

In Exercises 25–32, parametric equations for a curve are given. Find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5–12.

25. $x = t, \quad y = t^2$
26. $x = \sqrt{t}, \quad y = 5t + 2$
27. $x = t^2 - t, \quad y = t^2 + t$
28. $x = t^2 - 1, \quad y = t^3 - t$
29. $x = \sec t, \quad y = \tan t$ on $(-\pi/2, \pi/2)$
30. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$
31. $x = \cos t \sin(2t), \quad y = \sin t \sin(2t)$ on $[-\pi/2, \pi/2]$
32. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$

In Exercises 33–40, find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3\sin(2t), \quad y = 3\cos(2t)$ on $[0, \pi]$
34. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$
35. $x = 5t + 2, \quad y = 1 - 3t$ on $[-1, 1]$
36. $x = 2t^{3/2}, \quad y = 3t$ on $[0, 1]$
37. $x = \cos t, \quad y = \sin t$ on $[0, 2\pi]$
38. $x = 1 + 3t^2, \quad y = 4 + 2t^3$ on $[0, 1]$
39. $x = \dfrac{t}{1 + t}, \quad y = \ln(1 + t)$ on $[0, 2]$
40. $x = e^t - t, \quad y = 4e^{t/2}$ on $[-8, 3]$

In Exercises 41–44, numerically approximate the given arc length.

41. Approximate the arc length of one petal of the rose curve $x = \cos t \cos(2t), \quad y = \sin t \cos(2t)$ using Simpson's Rule and $n = 4$.
42. Approximate the arc length of the "bow tie curve" $x = \cos t, \quad y = \sin(2t)$ using Simpson's Rule and $n = 6$.
43. Approximate the arc length of the parabola $x = t^2 - t$, $y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.
44. A common approximate of the circumference of an ellipse given by $x = a\cos t, \quad y = b\sin t$ is $C \approx 2\pi\sqrt{\dfrac{a^2 + b^2}{2}}$. Use this formula to approximate the circumference of $x = 5\cos t, \quad y = 3\sin t$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

In Exercises 45–50, a solid of revolution is described. Find or approximate its surface area as specified.

45. Find the surface area of the sphere formed by rotating the circle $x = 2\cos t, \quad y = 2\sin t$ about:
    (a) the x-axis and
    (b) the y-axis.
46. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos t + 2, \quad y = \sin t$ about the y-axis.

47. Find the surface area of the solid formed by rotating the curve $x = a\cos^3\theta, \quad y = a\sin^3\theta$ on $[0, \pi/2]$ about the $x-$axis
48. Find the surface area of the solid formed by rotating the curve $x = 3t^2, \quad y = 2t^3$ on $[0, 5]$ about the $y-$axis
49. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos t$, $y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.
50. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos t\cos(2t)$, $y = \sin t\cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.
51. Find the area under the curve given by the parametric equations $x = \cosh t$, $y = \sinh t$, for $0 \leq t \leq \theta$. Subtract this area from the area of an appropriate triangle to verify the shaded area in the bottom graph of Figure 7.4.1.

10.4 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ).

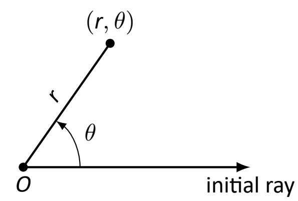


Figure 10.4.1: Illustrating polar coordinates.

Watch the video:
Polar Coordinates — The Basics at
`https://youtu.be/r0fv9V9GHdo`

Practice will make this process more clear.

Example 10.4.1 Plotting Polar Coordinates

Plot the following polar coordinates:

$$A(1, \pi/4) \quad B(1.5, \pi) \quad C(2, -\pi/3) \quad D(-1, \pi/4)$$

Solution To aid in the drawing, a polar grid is provided at the bottom of this page. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

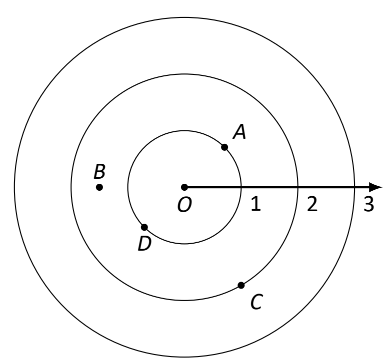
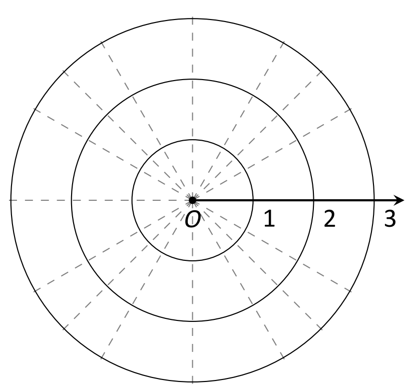


Figure 10.4.2: Plotting polar points in Example 10.4.1.

Notes:



To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units — in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 10.4.2.

Consider the following two points: $A(1,\pi)$ and $B(-1,0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C(1,3\pi)$, or $D(1,-\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

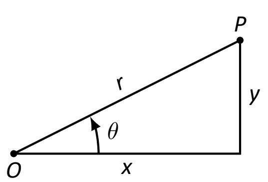


Figure 10.4.3: Converting between rectangular and polar coordinates.

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 10.4.3 shows a point P in the plane with rectangular coordinates (x,y) and polar coordinates (r,θ). Using trigonometry, we can make the identities given in the following Key Idea.

Key Idea 10.4.1 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r,\theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x,y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 10.4.2 Converting Between Polar and Rectangular Coordinates

Notes:

1. Convert the polar coordinates $A(2, 2\pi/3)$ and $B(-1, 5\pi/4)$ to rectangular coordinates.
2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

SOLUTION

1. (a) We start with $A(2, 2\pi/3)$. Using Key Idea 10.4.1, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $B(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 10.4.4 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \text{ radians} \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4.$$

Notes:

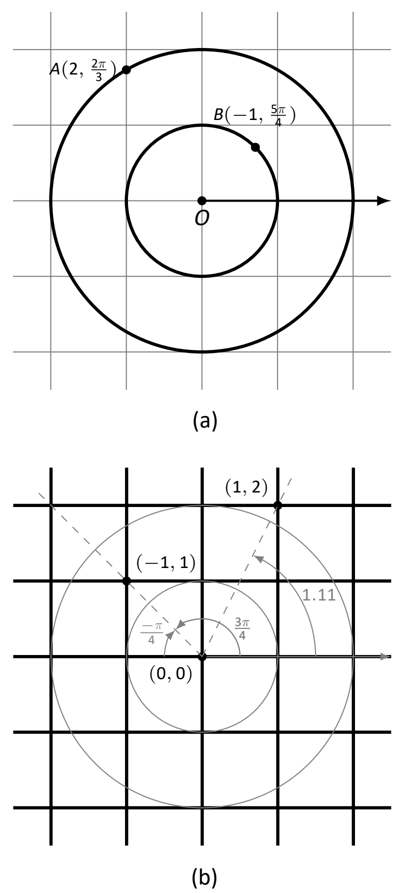


Figure 10.4.4: Plotting rectangular and polar points in Example 10.4.2.

This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 10.4.4 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 10.4.3 Introduction to Graphing Polar Functions

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 10.4.5.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45°$ with the initial ray.

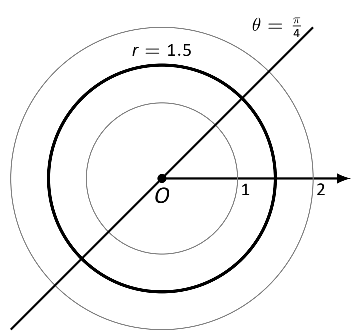


Figure 10.4.5: Plotting standard polar plots.

Notes:

We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

$$\tan(\pi/4) = y/x \quad \Rightarrow \quad x\tan(\pi/4) = y \quad \Rightarrow \quad y = x.$$

This graph is also plotted in Figure 10.4.5.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 10.4.4 Sketching Polar Functions

Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

Solution A common question when sketching curves by plotting selected points is "Which points should I plot?" With rectangular equations, we often chose "easy" values — integers, then added more if needed. When plotting polar equations, start with the "common" angles — multiples of $\pi/6$ and $\pi/4$. Figure 10.4.6 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $(2, 0)$ determined by the first line of the table. The angle is 0 radians — we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1 + \sqrt{3}/2$; so rotate by $\pi/6$ radians and go out $1 + \sqrt{3}/2$ units.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	$1 + \sqrt{3}/2$
$\pi/4$	$1 + 1/\sqrt{2}$
$\pi/3$	$3/2$
$\pi/2$	1
$2\pi/3$	$1/2$
$3\pi/4$	$1 - 1/\sqrt{2}$
$5\pi/6$	$1 - \sqrt{3}/2$
π	0
$7\pi/6$	$1 - \sqrt{3}/2$
$5\pi/4$	$1 - 1/\sqrt{2}$
$4\pi/3$	$1/2$
$3\pi/2$	1
$5\pi/3$	$3/2$
$7\pi/4$	$1 + 1/\sqrt{2}$
$11\pi/6$	$1 + \sqrt{3}/2$

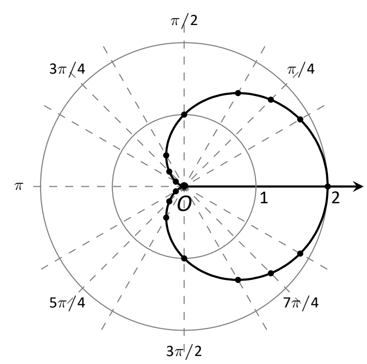


Figure 10.4.6: Graph of the polar function in Example 10.4.4 by plotting points.

Example 10.4.5 Sketching Polar Functions

Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 10.4.7. These points are then plotted in Figure 10.4.8. This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we

Notes:

numbered each point in the table and on the graph.

Pt.	θ	$\cos(2\theta)$	Pt.	θ	$\cos(2\theta)$
1	0	1	10	$7\pi/6$	0.5
2	$\pi/6$	0.5	11	$5\pi/4$	0
3	$\pi/4$	0	12	$4\pi/3$	-0.5
4	$\pi/3$	-0.5	13	$3\pi/2$	-1
5	$\pi/2$	-1	14	$5\pi/3$	-0.5
6	$2\pi/3$	-0.5	15	$7\pi/4$	0
7	$3\pi/4$	0	16	$11\pi/6$	0.5
8	$5\pi/6$	0.5	17	2π	1
9	π	1			

Figure 10.4.7: Tables of points for plotting a polar curve.

This plot is an example of a *rose curve*.

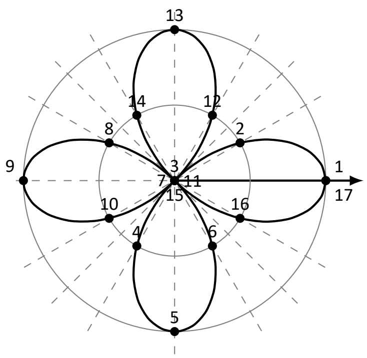


Figure 10.4.8: Polar plots from Example 10.4.5.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 10.4.1.

Example 10.4.6 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

SOLUTION

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

Notes:

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

   This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

   We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 10.4.9; note how it only exists in the first and third quadrants.

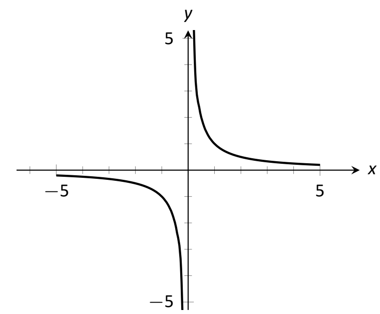


Figure 10.4.9: Graphing $xy = 1$ from Example 10.4.6.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

   The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

Notes:

We recognize this as a circle; by completing the square we can find its radius and center.

$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1.$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important to the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). Its rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as an exercise.

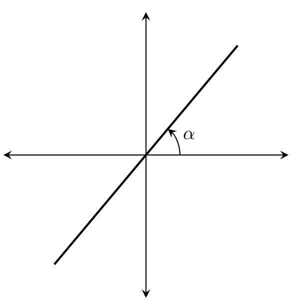
Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

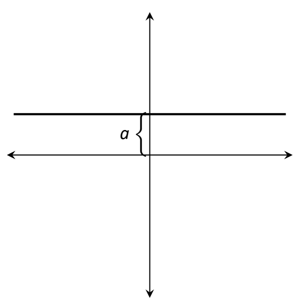
Through the origin:

$\theta = \alpha$



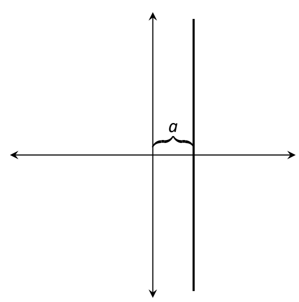
Horizontal line:

$r = a\csc\theta$



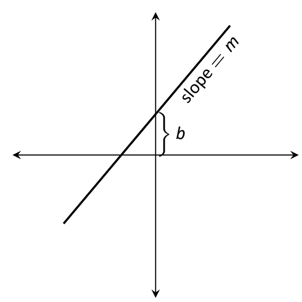
Vertical line:

$r = a\sec\theta$



Not through origin:

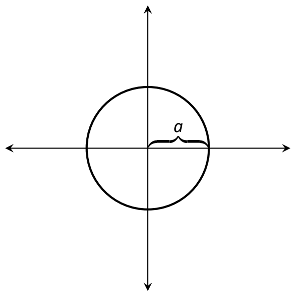
$$r = \frac{b}{\sin\theta - m\cos\theta}$$



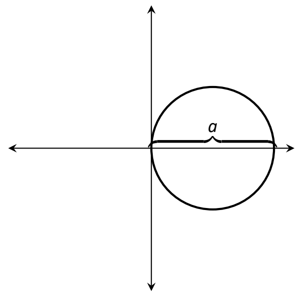
Notes:

Circles

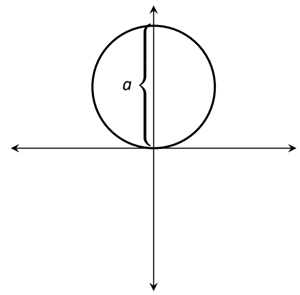
Centered on origin:
$r = a$



$(x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}$
$r = a\cos\theta$

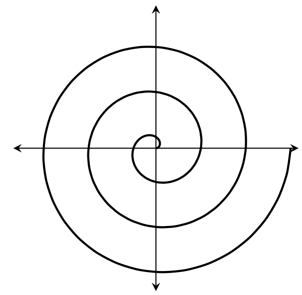


$x^2 + (y - \frac{a}{2})^2 = \frac{a^2}{4}$
$r = a\sin\theta$



Spiral

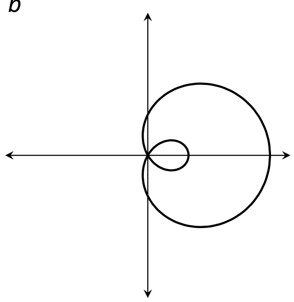
Archimedean spiral
$r = \theta$



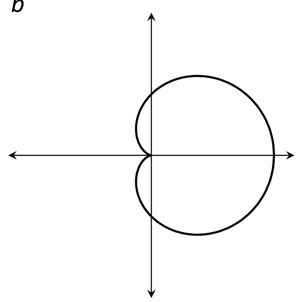
Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$; Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

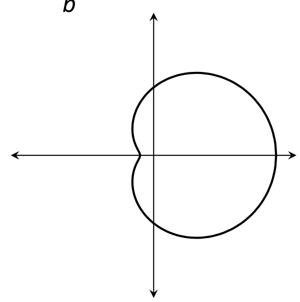
With inner loop:
$\frac{a}{b} < 1$



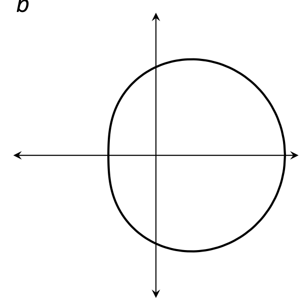
Cardioid:
$\frac{a}{b} = 1$



Dimpled:
$1 < \frac{a}{b} < 2$



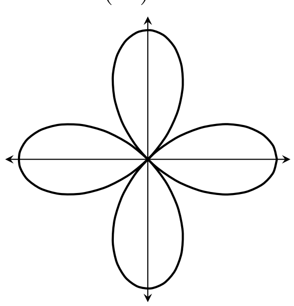
Convex:
$\frac{a}{b} > 2$



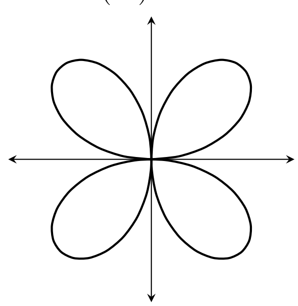
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$; Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

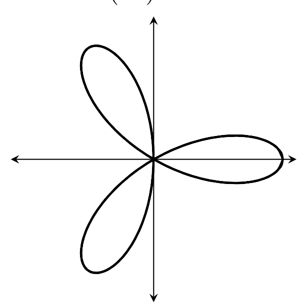
$r = a\cos(2\theta)$



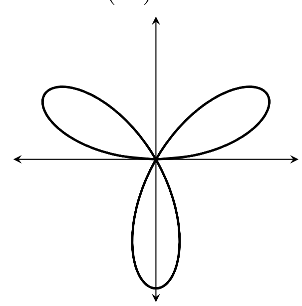
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



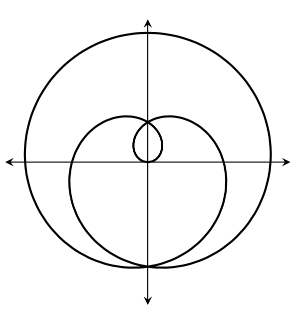
$r = a\sin(3\theta)$



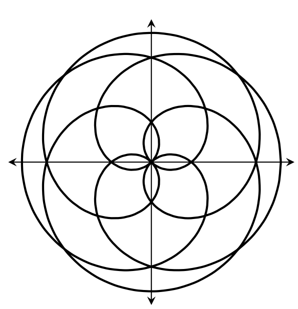
Special Curves

Rose curves

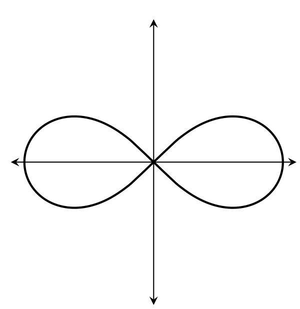
$r = a\sin(\theta/5)$



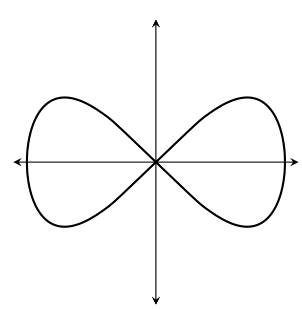
$r = a\sin(2\theta/5)$



Lemniscate:
$r^2 = a^2\cos(2\theta)$



Eight Curve:
$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 10.4.7 Finding points of intersection with polar curves
Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 10.4.10(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

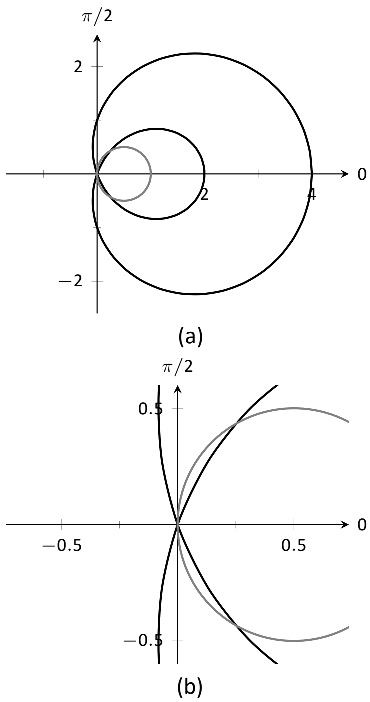


Figure 10.4.10: Graphs to help determine the points of intersection of the polar functions given in Example 10.4.7.

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4th quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 1st quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly traced).

Notes:

The limaçon intersects the pole when $1 + 3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106$ radians $\approx 109.47°$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 \approx 250.53°$.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Notes:

Exercises 10.4

Terms and Concepts

1. In your own words, describe how to plot the polar point $P(r, \theta)$.
2. T/F: When plotting a point with polar coordinate $P(r, \theta)$, r must be positive.
3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.
4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

5. Plot the points with the given polar coordinates.

   (a) $A(2, 0)$ (a) $C(-2, \pi/2)$

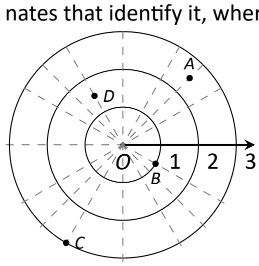
   (b) $B(1, \pi)$ (b) $D(1, \pi/4)$

6. Plot the points with the given polar coordinates.

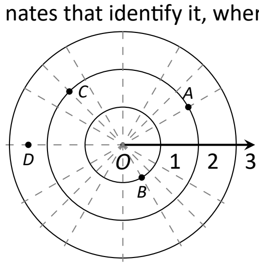
   (a) $A(2, 3\pi)$ (a) $C(1, 2)$

   (b) $B(1, -\pi)$ (b) $D(1/2, 5\pi/6)$

7. For each of the given points give two sets of polar coordinates that identify it, where $0 \leq \theta \leq 2\pi$.



8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \leq \theta \leq \pi$.



9. Convert the polar coordinates A and B to rectangular, and the rectangular coordinates C and D to polar.

   (a) $A(2, \pi/4)$ (a) $C(2, -1)$

   (b) $B(2, -\pi/4)$ (b) $D(-2, 1)$

10. Convert the polar coordinates A and B to rectangular, and the rectangular coordinates C and D to polar.

   (a) $A(3, \pi)$ (a) $C(0, 4)$

   (b) $B(1, 2\pi/3)$ (b) $D(1, -\sqrt{3})$

In Exercises 11–32, graph the polar function on the given interval.

11. $r = 2, \quad 0 \leq \theta \leq \pi/2$
12. $\theta = \pi/6, \quad -1 \leq r \leq 2$
13. $r = 1 - \cos\theta, \quad [0, 2\pi]$
14. $r = 2 + \sin\theta, \quad [0, 2\pi]$
15. $r = 2 - \sin\theta, \quad [0, 2\pi]$
16. $r = 1 - 2\sin\theta, \quad [0, 2\pi]$
17. $r = 1 + 2\sin\theta, \quad [0, 2\pi]$
18. $r = \cos(2\theta), \quad [0, 2\pi]$
19. $r = \sin(3\theta), \quad [0, \pi]$
20. $r = \cos(\theta/3), \quad [0, 3\pi]$
21. $r = \cos(2\theta/3), \quad [0, 6\pi]$
22. $r = \theta/2, \quad [0, 4\pi]$
23. $r = 3\sin(\theta), \quad [0, \pi]$
24. $r = -4\sin(\theta), \quad [0, \pi]$
25. $r = -2\cos(\theta), \quad [0, \pi]$
26. $r = \dfrac{3}{2}\cos(\theta), \quad [0, \pi]$
27. $r = \cos\theta\sin\theta, \quad [0, 2\pi]$
28. $r = \theta^2 - (\pi/2)^2, \quad [-\pi, \pi]$
29. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0, 2\pi]$
30. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0, 2\pi]$
31. $r = 3\sec\theta, \quad (-\pi/2, \pi/2)$
32. $r = 3\csc\theta, \quad (0, \pi)$

In Exercises 33–44, convert the polar equation to a rectangular equation.

33. $r = 2\cos\theta$
34. $r = -4\sin\theta$
35. $r = 3\sin\theta$
36. $r = -\dfrac{3}{2}\cos\theta$
37. $r = \cos\theta + \sin\theta$
38. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$
39. $r = \dfrac{3}{\cos\theta}$
40. $r = \dfrac{4}{\sin\theta}$
41. $r = \tan\theta$
42. $r = \cot\theta$
43. $r = 2$
44. $\theta = \dfrac{\pi}{6}$

In Exercises 45–52, convert the rectangular equation to a polar equation.

45. $y = x$
46. $y = 4x + 7$
47. $x = 5$
48. $y = 5$

49. $x = y^2$

50. $x^2 y = 1$

51. $x^2 + y^2 = 7$

52. $(x + 1)^2 + y^2 = 1$

In Exercises 53–60, find the points of intersection of the polar graphs.

53. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

54. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

55. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

56. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

57. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

58. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

59. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

60. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

61. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\frac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

62. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

10.5 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos\theta$ and $y = r\sin\theta$, we can create the parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and apply the concepts of Section 10.3.

Polar Functions and $\dfrac{dy}{dx}$

We are interested in the lines tangent to a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ to compute $\frac{dy}{dx}$.

Using Key Idea 10.3.1 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 10.5.1 Finding $\frac{dy}{dx}$ with Polar Functions

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Watch the video:
The Slope of Tangent Lines to Polar Curves at
`https://youtu.be/QTa9OZ4iGPo`

Notes:

Example 10.5.1 Finding $\frac{dy}{dx}$ with polar functions.
Consider the limaçon $r = 1 + 2\sin\theta$ on $[0, 2\pi]$.

1. Find the rectangular equations of the tangent and normal lines to the graph at $\theta = \pi/4$.
2. Find where the graph has vertical and horizontal tangent lines.

SOLUTION

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos\theta$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos\theta\sin\theta + \cos\theta(1 + 2\sin\theta)}{2\cos^2\theta - \sin\theta(1 + 2\sin\theta)}\\ &= \frac{\cos\theta(4\sin\theta + 1)}{2(\cos^2\theta - \sin^2\theta) - \sin\theta}.\end{aligned}$$

When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2} - 1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at $\theta = \pi/4$ is $(1 + \sqrt{2}/2, 1 + \sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2} - 1)\big(x - (1 + \sqrt{2}/2)\big) + 1 + \sqrt{2}/2 \approx -3.83x + 8.24.$$

The limaçon and the tangent line are graphed in Figure 10.5.1.

The normal line has the opposite-reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$ (when the denominator does not equal 0); thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos\theta(4\sin\theta + 1) = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \text{or} \quad 4\sin\theta + 1 = 0.$$

On $[0, 2\pi]$, $\cos\theta = 0$ when $\theta = \pi/2,\ 3\pi/2$.

Setting $4\sin\theta + 1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48^\circ$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the 3rd quadrant and one in the 4th. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 10.5.1 with black-filled dots.

Notes:

To find the vertical lines of tangency, we determine where $\frac{dy}{dx}$ is undefined by setting the denominator of $\frac{dy}{dx} = 0$ (when the numerator does not equal 0).

$$2(\cos^2\theta - \sin^2\theta) - \sin\theta = 0.$$

Convert the $\cos^2\theta$ term to $1 - \sin^2\theta$:

$$\begin{aligned} 2(1 - \sin^2\theta - \sin^2\theta) - \sin\theta &= 0 \\ 4\sin^2\theta + \sin\theta - 2 &= 0. \end{aligned}$$

Recognize this as a quadratic in the variable $\sin\theta$. Using the quadratic formula, we have

$$\sin\theta = \frac{-1 \pm \sqrt{33}}{8}.$$

We solve $\sin\theta = \frac{-1+\sqrt{33}}{8}$ and $\sin\theta = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin\theta &= \frac{-1+\sqrt{33}}{8} & \sin\theta &= \frac{-1-\sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) & \theta &= \sin^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) \\ \theta &\approx 0.6349 & \theta &\approx -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1} x$ only returns solutions in $[-\pi/2, \pi/2]$, the 4th and 1st quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1+\sqrt{33}}{8} \quad \Rightarrow \quad \theta \approx 0.6349,\ 2.5067 \text{ radians}$$

and

$$\sin\theta = \frac{-1-\sqrt{33}}{8} \quad \Rightarrow \quad \theta \approx 4.1446,\ 5.2802 \text{ radians}.$$

These points are also shown in Figure 10.5.1 with white-filled dots.

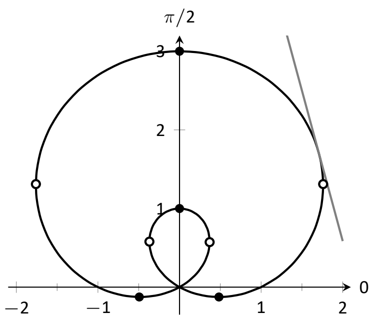


Figure 10.5.1: The limaçon in Example 10.5.1 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency.

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Making this substitution in the formula for $\frac{dy}{dx}$

Notes:

given in Key Idea 10.5.1 we see

$$\frac{dy}{dx} = \frac{f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha}{f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 10.4.3) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 10.5.2 Finding tangent lines at the pole

Let $r = 1 + 2\sin\theta$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

Solution We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin\theta &= 0 \\ \sin\theta &= -1/2 \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar coordinates, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x = \frac{x}{\sqrt{3}}$ and $y = \tan(11\pi/6)x = -\frac{x}{\sqrt{3}}$. The full limaçon can be seen in Figure 10.5.1; we zoom in on the tangent lines in Figure 10.5.2.

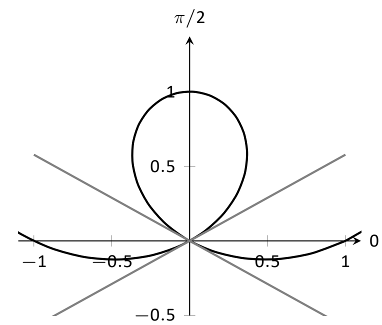


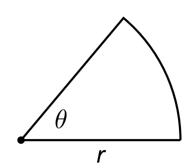
Figure 10.5.2: Graphing the tangent lines at the pole in Example 10.5.2.

Area

When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horizontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centered at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Note: Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



Consider Figure 10.5.3 (a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$,

Notes:

whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_0 < \theta_1 < \ldots < \theta_n = \beta$. The radian length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the i^{th} subinterval $[\theta_{i-1}, \theta_i]$ can be approximated with a sector of a circle with radius $f(c_i)$, for some c_i in $[\theta_{i-1}, \theta_i]$. The area of this sector is $\frac{1}{2}[f(c_i)]^2\Delta\theta$. This is shown in part (b) of the figure, where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2}[f(c_i)]^2\Delta\theta.$$

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the exact area of the region in the form of a definite integral.

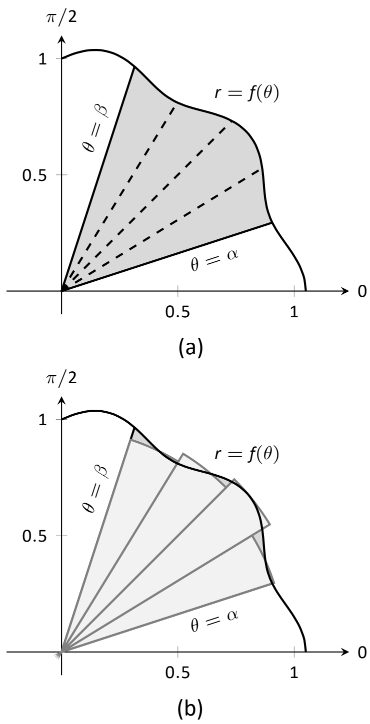


Figure 10.5.3: Computing the area of a polar region.

Theorem 10.5.1 Area of a Polar Region

Let f be continuous and non-negative on $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

The theorem states that $0 \leq \beta - \alpha \leq 2\pi$. This ensures that region does not overlap itself, which would give a result that does not correspond directly to the area.

Example 10.5.3 Area of a polar region

Find the area of the circle defined by $r = \cos\theta$. (Recall this circle has radius $1/2$.)

Solution This is a direct application of Theorem 10.5.1. The circle is

Notes:

traced out on $[0, \pi]$, leading to the integral

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^{\pi} \cos^2\theta\, d\theta \\ &= \frac{1}{2}\int_0^{\pi} \frac{1+\cos(2\theta)}{2}\, d\theta \\ &= \frac{1}{4}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_0^{\pi} \\ &= \frac{\pi}{4}.\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct.

Example 10.5.4 Area of a polar region

Find the area of the cardioid $r = 1+\cos\theta$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 10.5.4.

Solution This is again a direct application of Theorem 10.5.1.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos\theta)^2\, d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+2\cos\theta+\cos^2\theta)\, d\theta \\ &= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_{\pi/6}^{\pi/3} \\ &= \frac{1}{8}\big(\pi + 4\sqrt{3} - 4\big).\end{aligned}$$

Area Between Curves

Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 10.5.5. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha, \beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r_2^2\, d\theta - \frac{1}{2}\int_{\alpha}^{\beta} r_1^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \big(r_2^2 - r_1^2\big)\, d\theta.$$

Notes:

Note: Example 10.5.3 requires the use of the integral $\int \cos^2\theta\, d\theta$. This is handled well by using the half angle formula as found in the back of this text. Due to the nature of the area formula, integrating $\cos^2\theta$ and $\sin^2\theta$ is required often. We offer here these indefinite integrals as a time-saving measure.

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C \\ \int \sin^2\theta\, d\theta &= \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C\end{aligned}$$

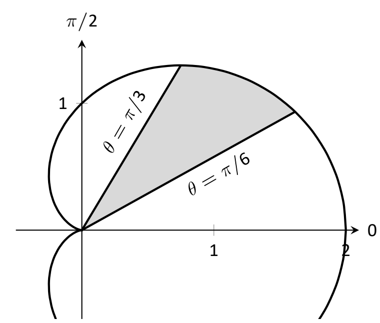


Figure 10.5.4: Finding the area of the shaded region of a cardioid in Example 10.5.4.

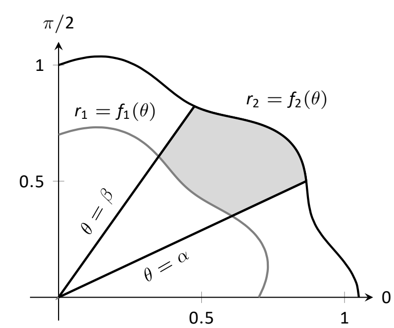


Figure 10.5.5: Illustrating area bound between two polar curves.

Key Idea 10.5.2 Area Between Polar Curves

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $0 \leq f_1(\theta) \leq f_2(\theta)$ on $[\alpha, \beta]$, is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f_2(\theta)]^2 - [f_1(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_2^2 - r_1^2\right) d\theta.$$

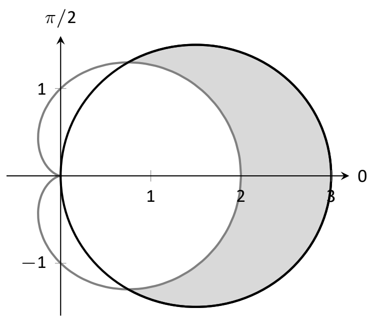


Figure 10.5.6: Finding the area between polar curves in Example 10.5.5.

Example 10.5.5 Area between polar curves

Find the area bounded between the curves $r = 1 + \cos\theta$ and $r = 3\cos\theta$, as shown in Figure 10.5.6.

Solution We need to find the points of intersection between these two functions. Setting them equal to each other, we find:

$$\begin{aligned} 1 + \cos\theta &= 3\cos\theta \\ \cos\theta &= 1/2 \\ \theta &= \pm\pi/3 \end{aligned}$$

Thus we integrate $\frac{1}{2}\left((3\cos\theta)^2 - (1+\cos\theta)^2\right)$ on $[-\pi/3, \pi/3]$.

$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left((3\cos\theta)^2 - (1+\cos\theta)^2\right) d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right) d\theta \\ &= \frac{1}{2}\left(2\sin(2\theta) - 2\sin\theta + 3\theta\right)\Big|_{-\pi/3}^{\pi/3} \\ &= \pi. \end{aligned}$$

Amazingly enough, the area between these curves has a "nice" value.

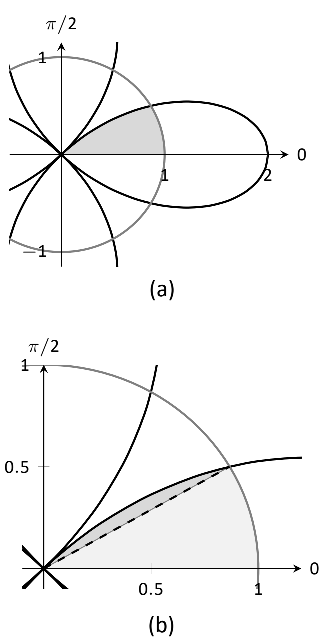


Figure 10.5.7: Graphing the region bounded by the functions in Example 10.5.6.

Example 10.5.6 Area defined by polar curves

Find the area bounded between the polar curves $r = 1$ and one petal of $r = 2\cos(2\theta)$ where $y > 0$, as shown in Figure 10.5.7(a).

Solution We need to find the point of intersection between the two

Notes:

curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta) = 1 \quad \Rightarrow \quad \cos(2\theta) = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \pi/3 \quad \Rightarrow \quad \theta = \pi/6.$$

In part (b) of the figure, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta = 0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r = 1$, $\theta = 0$ and $\theta = \pi/6$. (Note: the dashed line lies on the line $\theta = \pi/6$.) Above the dashed line the region is bounded by $r = 2\cos(2\theta)$ and $\theta = \pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

$$A_1 = \frac{1}{2}\int_0^{\pi/6} (1)^2\, d\theta \qquad A_2 = \frac{1}{2}\int_{\pi/6}^{\pi/4} \big(2\cos(2\theta)\big)^2\, d\theta.$$

(The upper bound of the integral computing A_2 is $\pi/4$ as $r = 2\cos(2\theta)$ is at the pole when $\theta = \pi/4$.)

We omit the integration details and let the reader verify that $A_1 = \pi/12$ and $A_2 = \pi/12 - \sqrt{3}/8$; the total area is $A = \pi/6 - \sqrt{3}/8$.

Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x = f(t)$, $y = g(t)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt. \qquad (10.5.1)$$

Now consider the polar function $r = f(\theta)$. We again use the identities $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (10.5.1).

The expression $[x'(\theta)]^2 + [y'(\theta)]^2$ can be simplified a great deal; we leave this as an exercise and state that

$$[x'(\theta)]^2 + [y'(\theta)]^2 = [f'(\theta)]^2 + [f(\theta)]^2.$$

This leads us to the arc length formula.

Notes:

Key Idea 10.5.3 Arc Length of Polar Curves

Let $r = f(\theta)$ be a polar function with f' continuous on an open interval I containing $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\, d\theta.$$

Example 10.5.7 Arc Length of Polar Curves

Find the arc length of the cardioid $r = 1 + \cos\theta$.

Solution With $r = 1 + \cos\theta$, we have $r' = -\sin\theta$. The cardioid is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Key Idea 10.5.3 we have

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{\sin^2\theta + (1 + 2\cos\theta + \cos^2\theta)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, \frac{\sqrt{2 - 2\cos\theta}}{\sqrt{2 - 2\cos\theta}}\, d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4 - 4\cos^2\theta}}{\sqrt{2 - 2\cos\theta}}\, d\theta \\
&= 2\int_0^{2\pi} \frac{\sqrt{1 - \cos^2\theta}}{\sqrt{2 - 2\cos\theta}}\, d\theta \\
&= 2\int_0^{2\pi} \frac{|\sin\theta|}{\sqrt{2 - 2\cos\theta}}\, d\theta
\end{aligned}$$

Since the $\sin\theta > 0$ on $[0, \pi]$ and $\sin\theta < 0$ on $[\pi, 2\pi]$ we separate the integral into two parts

$$2\int_0^{\pi} \frac{\sin\theta}{\sqrt{2 - 2\cos\theta}}\, d\theta - 2\int_\pi^{2\pi} \frac{\sin\theta}{\sqrt{2 - 2\cos\theta}}\, d\theta$$

Using the symmetry of the cardioid and u-substitution ($u = 2 - 2\cos\theta$) we

Notes:

simplify the integration to

$$\begin{aligned} L &= 4\int_0^{\pi} \frac{\sin\theta}{\sqrt{2-2\cos\theta}}\,d\theta \\ &= 2\int_0^4 \frac{1}{\sqrt{u}}\,du \\ &= 4u^{1/2}\Big|_0^4 = 8. \end{aligned}$$

Example 10.5.8 Arc length of a limaçon

Find the arc length of the limaçon $r = 1 + 2\sin\theta$.

Solution With $r = 1 + 2\sin\theta$, we have $r' = 2\cos\theta$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Key Idea 10.5.3, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1+2\sin\theta)^2}\,d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\,d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\,d\theta \\ &\approx 13.3649. \end{aligned}$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n = 4$, approximates the value with 13.0608. Using $n = 22$ gives the value above, which is accurate to 4 places after the decimal.)

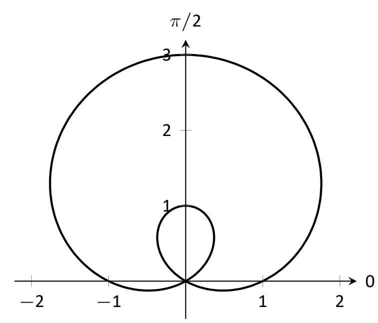


Figure 10.5.8: The limaçon in Example 10.5.8 whose arc length is measured.

Surface Area

The formula for arc length leads us to a formula for surface area. The following Key Idea is based on Key Idea 10.3.4.

Notes:

Key Idea 10.5.4 Surface Area of a Solid of Revolution

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on an open interval containing $[\alpha, \beta]$ on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\,d\theta.$$

2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\,d\theta.$$

Example 10.5.9 Surface area determined by a polar curve

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis (see Figure 10.5.9).

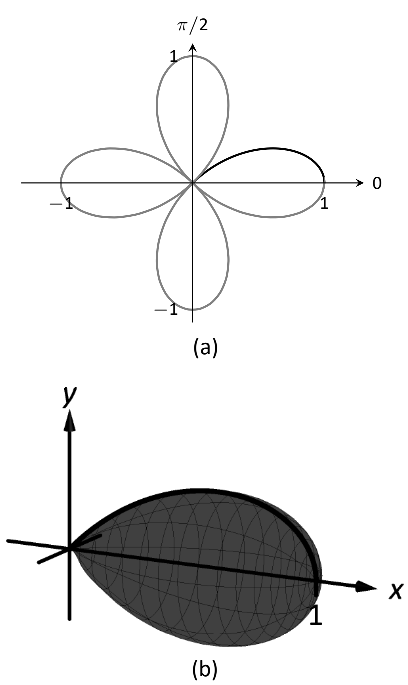


Figure 10.5.9: Finding the surface area of a rose-curve petal that is revolved around its central axis.

Solution We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Key Idea 10.5.4 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^{\pi/4} \cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\,d\theta \\ &\approx 1.36707.\end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D — that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

Notes:

Exercises 10.5

Terms and Concepts

1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?
2. With rectangular coordinates, it is natural to approximate area with ________; with polar coordinates, it is natural to approximate area with ________.

Problems

In Exercises 3–10, find:

(a) $\dfrac{dy}{dx}$

(b) the equation of the tangent and normal lines to the curve at the indicated θ-value.

3. $r = 1; \quad \theta = \pi/4$
4. $r = \cos\theta; \quad \theta = \pi/4$
5. $r = 1 + \sin\theta; \quad \theta = \pi/6$
6. $r = 1 - 3\cos\theta; \quad \theta = 3\pi/4$
7. $r = \theta; \quad \theta = \pi/2$
8. $r = \cos(3\theta); \quad \theta = \pi/6$
9. $r = \sin(4\theta); \quad \theta = \pi/3$
10. $r = \dfrac{1}{\sin\theta - \cos\theta}; \quad \theta = \pi$

In Exercises 11–14, find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.

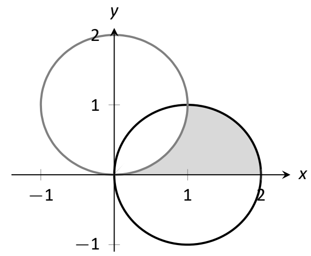
11. $r = 3; \quad [0, 2\pi]$
12. $r = 2\sin\theta; \quad [0, \pi]$
13. $r = \cos(2\theta); \quad [0, 2\pi]$
14. $r = 1 + \cos\theta; \quad [0, 2\pi]$

In Exercises 15–18, find the equation of the lines tangent to the graph at the pole.

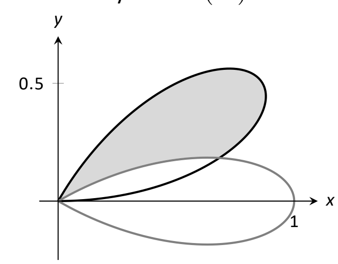
15. $r = \sin\theta; \quad [0, \pi]$
16. $r = \cos 3\theta; \quad [0, \pi]$
17. $r = \cos 2\theta; \quad [0, 2\pi]$
18. $r = \sin 2\theta; \quad [0, 2\pi]$

In Exercises 19–30, find the area of the described region.

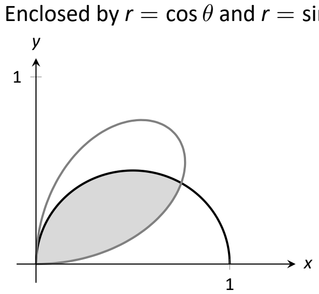
19. Enclosed by the circle: $r = 4\sin\theta, \quad \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$
20. Enclosed by the circle $r = 5$
21. Enclosed by one petal of $r = \sin(3\theta)$
22. Enclosed by one petal of the rose curve $r = \cos(n\ \theta)$, where n is a positive integer.
23. Enclosed by the cardioid $r = 1 - \sin\theta$
24. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$
25. Enclosed by the outer loop of the limaçon $r = 1 + 2\cos\theta$ (including area enclosed by the inner loop)
26. Enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos\theta$
27. Enclosed by $r = 2\cos\theta$ and $r = 2\sin\theta$, as shown:



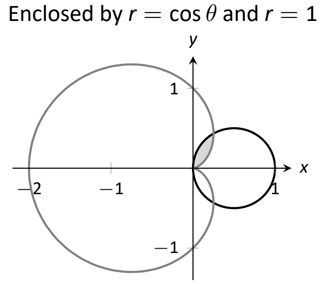
28. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



29. Enclosed by $r = \cos\theta$ and $r = \sin(2\theta)$, as shown:



30. Enclosed by $r = \cos\theta$ and $r = 1 - \cos\theta$, as shown:



In Exercises 31–36, answer the questions involving arc length.

31. Let $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$. Show, as suggested by the text, that
$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$
32. Use the arc length formula to compute the arc length of the circle $r = 2$.
33. Use the arc length formula to compute the arc length of the circle $r = 4\sin\theta$.
34. Use the arc length formula to compute the arc length of $r = \cos\theta + \sin\theta$.
35. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.
36. Approximate the arc length of the cardioid $r = 1 + \cos\theta$ with Simpson's Rule and $n = 6$.

In Exercises 37–42, answer the questions involving surface area.

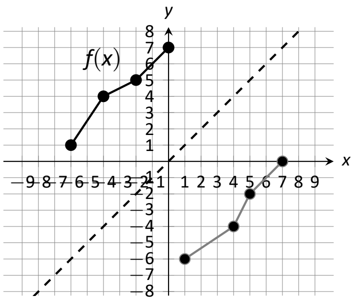
37. Use Key Idea 10.5.4 to find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.
38. Use Key Idea 10.5.4 to find the surface area of the sphere formed by revolving the circle $r = 2\cos\theta$ about the initial ray.
39. Find the surface area of the solid formed by revolving the cardioid $r = 1 + \cos\theta$ about the initial ray.
40. Find the surface area of the solid formed by revolving the circle $r = 2\cos\theta$ about the line $\theta = \pi/2$.
41. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $-\pi/4 \leq \theta \leq \pi/4$, about the line $\theta = \pi/2$.
42. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $0 \leq \theta \leq \pi/4$, about the initial ray.

Appendix

Solutions To Selected Problems

Chapter 7

Exercises 7.1

1. F
3. The point $(10, 1)$ lies on the graph of $y = f^{-1}(x)$ (assuming f is invertible).
5. 

7. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.
9. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.
11. $[-4, 0]$ or $[0, 4]$
13. $(-\infty, 3]$ or $[3, \infty)$
15. $f^{-1}(x) = \dfrac{2x+1}{x-1}$
17. $f^{-1}(x) = \ln(x+2) - 3$
19. 0
21. $-1/5$
23. $1/\sqrt{2}$
25. $7/\sqrt{58}$
27. $3\pi/4$
29. $x/2$
31. $x/\sqrt{x^2+25}$
33.
35.
37. $2\pi/3\sqrt{3}$

Exercises 7.2

1. The point $(10, 1)$ lies on the graph of $y = f^{-1}(x)$ (assuming f is invertible) and $(f^{-1})'(10) = 1/5$.
3. $\left(f^{-1}\right)'(20) = \frac{1}{f'(2)} = 1/5$
5. $\left(f^{-1}\right)'(\sqrt{3}/2) = \frac{1}{f'(\pi/6)} = 1$
7. $\left(f^{-1}\right)'(1/2) = \frac{1}{f'(1)} = -2$
9. $h'(t) = \frac{2}{\sqrt{1-4t^2}}$
11. $g'(x) = \frac{2}{1+4x^2}$
13. $g'(t) = \cos^{-1}(t)\cos(t) - \frac{\sin(t)}{\sqrt{1-t^2}}$
15. $h'(x) = \frac{\sin^{-1}x + \cos^{-1}x}{\sqrt{1-x^2}(\cos^{-1}x)^2}$
17. $f'(x) = -\frac{1}{\sqrt{1-x^2}}$
19. (a) $f(x) = x$, so $f'(x) = 1$
    (b) $f'(x) = \cos(\sin^{-1}x)\frac{1}{\sqrt{1-x^2}} = 1.$
21. (a) $f(x) = \sqrt{1-x^2}$, so $f'(x) = \frac{-x}{\sqrt{1-x^2}}$
    (b) $f'(x) = \cos(\cos^{-1}x)\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{-x}{\sqrt{1-x^2}}$
23. $y = \sqrt{2}(x - \sqrt{2}/2) + \pi/4$
25. $-\pi/6$
27. $\frac{1}{2}\left(\sin^{-1}r\right)^2 + C$
29. $\sin^{-1}(e^t/\sqrt{10}) + C$
31. $\sqrt{91} \approx 9.54$ feet

Exercises 7.3

1. $(-\infty, \infty)$
3. $(-\infty, 0) \cup (0, \infty)$
5. $f'(t) = 3t^2e^{t^3-1}$
7. $f'(x) = \dfrac{1 - x\ln 5 \ln x}{x5^x \ln 5}$
9. $f'(x) = 1$
11. $h'(r) = \dfrac{3^r \ln 3}{1+3^{2r}}$
13. $\dfrac{24}{\ln 5}$
15. $\frac{3^{x^2-1}}{2ln3} + C$

17. $\frac{1}{2}\sin^2(e^x)+C$

19. $\dfrac{\ln 245}{\ln 3}-1$

21. $\frac{1}{2}\ln^2(x)+C$

23. $\frac{1}{6}\ln^2\left(x^3\right)+C$

25. $n=-3,2$

27. $y'=(1+x)^{1/x}\left(\frac{1}{x(x+1)}-\frac{\ln(1+x)}{x^2}\right)$
Tangent line: $y=(1-2\ln 2)(x-1)+2$

29. $y'=\frac{x^x}{x+1}\left(\ln x+1-\frac{1}{x+1}\right)$
Tangent line: $y=(1/4)(x-1)+1/2$

31. $y'=\frac{x+1}{x+2}\left(\frac{1}{x+1}-\frac{1}{x+2}\right)$
Tangent line: $y=1/9(x-1)+2/3$

33. $y'=x^{e^x-1}e^x(1+x\ln x)$
Tangent line: $y=ex-e+1$

35. $r=(\ln 2)/5730$; $5730\ln 10/\ln 2\approx 19034.65$ years

Exercises 7.4

1. Because $\cosh x$ is always positive.

3. $\cosh t=13/12$, etc.

5.
$$\begin{aligned}\coth^2 x-\operatorname{csch}^2 x&=\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2-\left(\frac{2}{e^x-e^{-x}}\right)^2\\&=\frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\&=\frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\&=1\end{aligned}$$

7.
$$\begin{aligned}\cosh^2 x&=\left(\frac{e^x+e^{-x}}{2}\right)^2\\&=\frac{e^{2x}+2+e^{-2x}}{4}\\&=\frac{1}{2}\frac{(e^{2x}+e^{-2x})+2}{2}\\&=\frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\&=\frac{\cosh 2x+1}{2}.\end{aligned}$$

9.
$$\begin{aligned}\frac{d}{dx}[\operatorname{sech} x]&=\frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\&=\frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\&=-\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\&=-\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\&=-\operatorname{sech} x\tanh x\end{aligned}$$

11. $\displaystyle\int\tanh x\,dx=\int\frac{\sinh x}{\cosh x}\,dx$
Let $u=\cosh x$; $du=(\sinh x)\,dx$
$$\begin{aligned}&=\int\frac{1}{u}\,du\\&=\ln|u|+C\\&=\ln(\cosh x)+C.\end{aligned}$$

13. $2\cosh 2x$

15. $2x\operatorname{sech}^2(x^2)$

17. $\sinh^2 x+\cosh^2 x$

19. $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

21. $\frac{4x}{\sqrt{4x^4-1}}$

23. $-\csc x$

25. $y=x$

27. $y=\frac{9}{25}(x+\ln 3)-\frac{4}{5}$

29. $y=x$

31. $\frac{1}{2}\ln(\cosh(2x))+C$

33. $\frac{1}{2}\sinh^2 x+C$ or $1/2\cosh^2 x+C$

35. $\cosh^{-1}(x^2/2)+C=\ln(x^2+\sqrt{x^4-4})+C$

37. $\tan^{-1}(e^x)+C$

39. 0

41. Using rule #32.: $A=\int_0^{\sinh\theta}\sqrt{1+y^2}-y\coth\theta\,dy=\frac{\theta}{2}$.

Exercises 7.5

1. $0/0,\infty/\infty,0\cdot\infty,\infty-\infty,0^0,1^\infty,\infty^0$

3. F

5. derivatives; limits

7. Answers will vary.

9. 3

11. -1

13. 5

15. a/b

17. $1/2$

19. 0

21. ∞

23. 0

25. -2

27. 0

29. 0

31. ∞

33. ∞

35. 0

37. 1

39. 1

41. 1

43. 1

45. 1

47. 2

49. $-\infty$

51. 0

53. $\sqrt[3]{5}$

55. Use technology to verify sketch.

Chapter 8

Exercises 8.1

1. T
3. $\sin x - x\cos x + C$
5. $-x^2\cos x + 2x\sin x + 2\cos x + C$
7. $1/2e^{x^2} + C$
9. $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$
11. $1/5e^{2x}(\sin x + 2\cos x) + C$
13. $\frac{1}{10}e^{5x}(\sin(5x) + \cos(5x)) + C$
15. $\sqrt{1-x^2} + x\sin^{-1}(x) + C$
17. $\frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$
19. $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$
21. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x-1| - \frac{x}{2} - \frac{1}{2}\ln|x-1| + C$
23. $\frac{1}{3}x^3\ln|x| - \frac{x^3}{9} + C$
25. $2x + x\left(\ln|x+1|\right)^2 + \left(\ln|x+1|\right)^2 - 2x\ln|x+1| - 2\ln|x+1| + 2 + C$
27. $\ln|\sin x| - x\cot x + C$
29. $\frac{1}{3}(x^2-2)^{3/2} + C$
31. $x\sec x - \ln|\sec x + \tan x| + C$
33. $x\sinh x - \cosh x + C$
35. $x\sinh^{-1}x - \sqrt{x^2+1} + C$
37. $1/2x\big(\sin(\ln x) - \cos(\ln x)\big) + C$
39. $\frac{1}{2}x\ln|x| - \frac{x}{2} + C$
41. $1/2x^2 + C$
43. π
45. 0
47. $1/2$
49. $\frac{3}{4e^2} - \frac{5}{4e^4}$
51. $\frac{1}{5}\left(e^{\pi} + e^{-\pi}\right)$
53.
55. (a) $b_n = (-1)^{n+1}2/n$
    (b) answers will vary

Exercises 8.2

1. F
3. F
5. $\frac{1}{4}\sin^4 x + C$
7. $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$
9. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + C$
11. $\frac{1}{2}\cos^2 x - \ln|\cos x| + C$
13. $\left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$
15. $\frac{1}{2}\left(\frac{1}{4}\sin(4x) - \frac{1}{10}\sin(10x)\right) + C$
17. $\frac{1}{2}\left(\sin(x) + \frac{1}{3}\sin(3x)\right) + C$
19. $\tan x - x + C$
21. $\frac{\tan^6(x)}{6} + \frac{\tan^4 x}{4} + C$
23. $\frac{\sec^5(x)}{5} - \frac{\sec^3 x}{3} + C$
25. $\frac{1}{3}\tan^3 x - \tan x + x + C$
27. $\frac{1}{2}\left(\sec x\tan x - \ln|\sec x + \tan x|\right) + C$
29. $\ln|\csc x - \cot x| + C$
31. $-\frac{1}{2}\cot^2 x + \ln|\csc x| + C$
33. $\frac{2}{5}$
35. $32/315$
37. $2/3$
39. $16/15$
41. 1

Exercises 8.3

1. backwards
3. (a) $\tan^2\theta + 1 = \sec^2\theta$
   (b) $9\sec^2\theta$.
5. $\frac{1}{2}\left(x\sqrt{x^2+1} + \ln\left|\sqrt{x^2+1} + x\right|\right) + C$
7. $x\sqrt{x^2+1/4} + \frac{1}{4}\ln\left|2\sqrt{x^2+1/4} + 2x\right| + C = \frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln\left|\sqrt{4x^2+1} + 2x\right| + C$
9. $4\left(\frac{1}{2}x\sqrt{x^2-1/16} - \frac{1}{32}\ln\left|4x + 4\sqrt{x^2-1/16}\right|\right) + C = \frac{1}{2}x\sqrt{16x^2-1} - \frac{1}{8}\ln\left|4x + \sqrt{16x^2-1}\right| + C$
11. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$ (Trig. Subst. is not needed)
13. $2\left(\frac{x}{4}\sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+1}}{2} + \frac{x}{2}\right|\right) + C$
15. $\frac{1}{2}\left(9\sin^{-1}(x/3) + x\sqrt{9-x^2}\right) + C$
17. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$
19. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$
21. $\frac{5}{4}\sec^{-1}(|x|/4) + C$
23. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}} + C$
25. $3\sin^{-1}\left(\frac{x-4}{5}\right) + C$
27. $\sqrt{x^2-11} - \sqrt{11}\sec^{-1}(x/\sqrt{11}) + C$
29. $-\frac{1}{\sqrt{x^2+9}} + C$ (Trig. Subst. is not needed)
31. $\frac{1}{18}\frac{x+2}{x^2+4x+13} + \frac{1}{54}\tan^{-1}\left(\frac{x+2}{3}\right) + C$
33. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x} - \sin^{-1}(x/\sqrt{5})\right) + C$
35. $\pi/2$
37. $2\sqrt{2} + 2\ln(1+\sqrt{2})$

39. $9\sin^{-1}(1/3) + 2\sqrt{2}$ Note: the new bounds of integration are $\sin^{-1}(-1/3) < \theta < \sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right) = \cos\left(\sin^{-1}(-1/3)\right) = 2\sqrt{2}/3$.

41. (a) $\pi(1 - \frac{\pi}{4})$

    (b) $\pi(\sqrt{2} - \ln(1 + \sqrt{2}))$

Exercises 8.4

1. rational
3. $\frac{A}{x} + \frac{B}{x-3}$
5. $\frac{A}{x-\sqrt{7}} + \frac{B}{x+\sqrt{7}}$
7. $3\ln|x-2| + 4\ln|x+5| + C$
9. $\frac{1}{3}(\ln|x+2| - \ln|x-2|) + C$
11. $-\frac{4}{x+8} - 3\ln|x+8| + C$
13. $-\ln|2x-3| + 5\ln|x-1| + 2\ln|x+3| + C$
15. $x + \ln|x-1| - \ln|x+2| + C$
17. $2x + C$
19. $\frac{1}{x} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$
21. $\ln\left|3x^2 + 5x - 1\right| + 2\ln|x+1| + C$
23. $\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x - \frac{1}{2(x^2+1)} + C$
25. $\frac{1}{2}\left(3\ln\left|x^2 + 2x + 17\right| - 4\ln|x-7| + \tan^{-1}\left(\frac{x+1}{4}\right)\right) + C$
27. $-\frac{1}{4}\ln(x^2+3) + \frac{1}{4}\ln(x^2+1) + C = \frac{1}{4}\ln\frac{x^2+1}{x^2+3} + C$
29. $3\left(\ln\left|x^2 - 2x + 11\right| + \ln|x-9|\right) + 3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right) + C$
31. $\frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{16}\tan^{-1}(x/2) + C$
33. $\ln x - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\frac{1}{x^2+1} + C$
35. $\ln(2000/243) \approx 2.108$
37. $-\pi/4 + \tan^{-1}3 - \ln(11/9) \approx 0.263$
39.

Exercises 8.5

1. $x\sin^{-1}x + \sqrt{1-x^2} + C$
3. $18\ln|x-2| - 9\ln|x-1| - 5\ln|x-3| + C$
5. $\dfrac{x}{25\sqrt{x^2+25}} + C$
7. $2\ln|x-1| - \ln|x| - \dfrac{1}{x-1} - \dfrac{1}{(x-1)^2} + C$
9. $\dfrac{1}{2}e^{x^2}(x^2-1) + C$
11. $\dfrac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C$
13. $-\sqrt{4-x^2} + C$
15. $2\tan^{-1}\sqrt{x} + C$
17. $\dfrac{1}{27}[6x\sin 3x - (9x^2-2)\cos 3x] + C$
19. $\dfrac{2}{3}(1+e^x)^{3/2} + C$
21. $\dfrac{1}{3}\tan^3 x + C$
23. $-\dfrac{1}{4}(8-x^3)^{4/3} + C$
25. $\dfrac{1}{10}(3-2x)^{5/2} - \dfrac{1}{2}(3-2x)^{3/2} + C$
27. $\dfrac{2}{5}x^{5/2} - \dfrac{8}{3}x^{3/2} + 6x^{1/2} + C$
29. $\dfrac{11}{2}\ln|x+5| - \dfrac{15}{2}\ln|x+7| + C$
31. $e^{\tan x} + C$
33. $-\dfrac{1}{5}\cot^5 x + \dfrac{1}{3}\cot^3 x - \cot x - x + C$
35. $\dfrac{1}{3}x^3 - \dfrac{1}{4}\tanh 4x + C$
37. $3\sin^{-1}\left(\dfrac{x+5}{6}\right) + C$
39. $\dfrac{1}{3}\sec^3 x - \sec x + C$
41. $-2\sin^{-1}\left(\dfrac{2x}{3}\right) - \dfrac{1}{x}\sqrt{9-4x^2} + C$
43. $-\ln x + \dfrac{4}{\sqrt[4]{x}} + 4\ln\left|1 - \sqrt[4]{x}\right| + C$
45. $\dfrac{-x}{2(25+x^2)} + \dfrac{1}{10}\tan^{-1}\left(\dfrac{x}{5}\right) + C$
47. $\dfrac{1}{4}x^4 - 2x^2 + 4\ln|x| + C$
49. $\dfrac{3}{64}(2x+3)^{8/3} - \dfrac{9}{20}(2x+3)^{5/3} + \dfrac{27}{16}(2x+3)^{2/3} + C$
51. $-\dfrac{1}{7}\cos 7x + C$
53.
55. $\frac{1}{2}\ln\left|\tan\frac{\theta}{2}\right| - \frac{1}{4}\tan^2\frac{\theta}{2} + C.$

Exercises 8.6

1. The interval of integration is finite, and the integrand is continuous on that interval.
3. converges; could also state ≤ 10.
5. $p > 1$
7. $e^5/2$
9. $1/3$
11. $1/\ln 2$
13. diverges
15. 1
17. diverges
19. diverges
21. $2\sqrt{3}$
23. diverges
25. diverges
27. 1
29. 0
31. $-1/4$

33. -1
35. diverges
37. diverges; Limit Comparison Test with $1/x$.
39. diverges; Limit Comparison Test with $1/x$.
41. converges; Direct Comparison Test with e^{-x}.
43. converges; Direct Comparison Test with $1/(x^2-1)$.
45. converges; Direct Comparison Test with $1/e^x$.
47. (a) $e^{-\lambda a}$
 (b) $\frac{1}{\lambda}$
 (c) e^{-1}

Exercises 8.7

1. F
3. They are superseded by the Trapezoidal Rule; it takes an equal amount of work and is generally more accurate.
5. (a) $3/4$
 (b) $2/3$
 (c) $2/3$
7. (a) $\frac{1}{4}(1+\sqrt{2})\pi \approx 1.896$
 (b) $\frac{1}{6}(1+2\sqrt{2})\pi \approx 2.005$
 (c) 2
9. (a) 38.5781
 (b) $147/4 \approx 36.75$
 (c) $147/4 \approx 36.75$
11. (a) 0
 (b) 0
 (c) 0
13. Trapezoidal Rule: 0.9006
 Simpson's Rule: 0.90452
15. Trapezoidal Rule: 13.9604
 Simpson's Rule: 13.9066
17. Trapezoidal Rule: 1.1703
 Simpson's Rule: 1.1873
19. Trapezoidal Rule: 1.0803
 Simpson's Rule: 1.077
21. (a) $n = 161$ (using max $\big(f''(x)\big) = 1$)
 (b) $n = 12$ (using max $\big(f^{(4)}(x)\big) = 1$)
23. (a) $n = 1004$ (using max $\big(f''(x)\big) = 39$)
 (b) $n = 62$ (using max $\big(f^{(4)}(x)\big) = 800$)
25. (a) Area is 30.8667 cm^2.
 (b) Area is $308,667$ yd^2.
27. Let $f(x) = a(x-x_1)^2 + b(x-x_1) + c$, so that $f(x_1) = c = y_1$, $f(x_1+\Delta x) = a\Delta x^2 + b\Delta x + c = y_2$, and $f(x_1+2\Delta x) = 4a\Delta x^2 + 2b\Delta x + c = y_3$. Therefore, $a = \frac{y_1-2y_2+y_3}{2(\Delta x)^2}$ and $b = \frac{4y_2-y_3-3y_1}{2\Delta x}$, and
$$\int_{x_1}^{x_1+2\Delta x} a(x-x_1)^2 + b(x-x_1) + c\,dx = \frac{a(2\Delta x)^3}{3} + \frac{b(2\Delta x)^2}{2} + c(2\Delta x) = \frac{4(y_1-2y_2+y_3)\Delta x}{3} + (4y_2-y_3-3y_1)\Delta x + 2y_1\Delta x = \frac{\Delta x}{3}(4y_1-8y_2+4y_3+12y_2-3y_3-9y_1+6y_1) = \frac{\Delta x}{3}(y_1+4y_2+y_3).$$

Chapter 9

Exercises 9.1

1. Answers will vary.
3. Answers will vary.
5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$
7. $-\frac{1}{3}, -2, -\frac{81}{5}, -\frac{512}{3}, -\frac{15625}{7}$
9. $a_n = 3n+1$
11. $a_n = 10\cdot 2^{n-1}$
13. $1/7$
15. 0
17. diverges
19. converges to 0
21. converges to 0
23. diverges
25. converges to e
27. converges to 5
29. diverges
31. converges to 0
33. converges to 0
35. converges to ln 2
37. converges to 0
39. bounded
41. bounded below
43. monotonically increasing
45. never monotonic
47. never monotonic
49. Let $\{a_n\}$ be given such that $\lim\limits_{n\to\infty} |a_n| = 0$. By the definition of the limit of a sequence, given any $\varepsilon > 0$, there is a m such that for all $n > m$, $|\,|a_n| - 0| < \varepsilon$. Since $|\,|a_n| - 0| = |a_n - 0|$, this directly implies that for all $n > m$, $|a_n - 0| < \varepsilon$, meaning that $\lim\limits_{n\to\infty} a_n = 0$.
51. Left to reader
53. (d) 2

Exercises 9.2

1. Answers will vary.
3. One sequence is the sequence of terms $\{a_n\}$. The other is the sequence of n^{th} partial sums, $\{S_n\} = \{\sum_{i=1}^{n} a_i\}$.
5. F
7. (a) $-1, -\frac{1}{2}, -\frac{5}{6}, -\frac{7}{12}, -\frac{47}{60}$
   (b) Plot omitted
9. (a) $-1, 0, -1, 0, -1$
   (b) Plot omitted
11. (a) $1, \frac{3}{2}, \frac{5}{3}, \frac{41}{24}, \frac{103}{60}$
   (b) Plot omitted
13. (a) $-0.9, -0.09, -0.819, -0.1629, -0.75339$
   (b) Plot omitted
15. Converges because it is a geometric series with $r = \frac{1}{5}$.
17. Diverges by Theorem 9.2.4
19. Diverges
21. $\lim\limits_{n\to\infty} a_n = 1$; by Theorem 9.2.4 the series diverges.
23. Diverges
25. Diverges
27. $\lim\limits_{n\to\infty} a_n = e$; by Theorem 9.2.4 the series diverges.
29. Converges
31. Converges
33. (a) $S_n = \frac{1-(1/4)^n}{3/4}$
   (b) Converges to $4/3$.
35. (a) $S_n = \begin{cases} -\frac{n+1}{2} & \text{n is odd} \\ \frac{n}{2} & \text{n is even} \end{cases}$
   (b) Diverges
37. (a) $S_n = \frac{1-(1/e)^{n+1}}{1-1/e}$.
   (b) Converges to $1/(1-1/e) = e/(e-1)$.
39. (a) With partial fractions, $a_n = \frac{1}{n} - \frac{1}{n+1}$. Thus $S_n = 1 - \frac{1}{n+1}$.
   (b) Converges to 1.
41. (a) Use partial fraction decomposition to recognize the telescoping series: $a_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$, so that $S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}$.
   (b) Converges to $1/2$.
43. (a) $S_n = 1 - \frac{1}{(n+1)^2}$
   (b) Converges to 1.
45. (a) $a_n = 1/2^n + 1/3^n$ for $n \geq 0$. Thus $S_n = \frac{1-1/2^2}{1/2} + \frac{1-1/3^n}{2/3}$.
   (b) Converges to $2 + 3/2 = 7/2$.
47. (a) $S_n = \frac{1-(\sin 1)^{n+1}}{1-\sin 1}$
   (b) Converges to $\frac{1}{1-\sin 1}$.
49. $(-3, 3)$
51. $(-\infty, -4) \cup (4, \infty)$
53. $\lim\limits_{n\to\infty} a_n = 3$; by Theorem 9.2.4 the series diverges.
55. $\lim\limits_{n\to\infty} a_n = \infty$; by Theorem 9.2.4 the series diverges.
57. $\lim\limits_{n\to\infty} a_n = 1/2$; by Theorem 9.2.4 the series diverges.
59. Using partial fractions, we can show that $a_n = \frac{1}{4}\left(\frac{1}{2n-1} + \frac{1}{2n+1}\right)$. The series is effectively twice the sum of the odd terms of the Harmonic Series which was shown to diverge in Example 9.2.5. Thus this series diverges.
61. $2\left(\frac{1+r}{1-r}\right)$

Exercises 9.3

1. continuous, positive and decreasing
3. Converges
5. Diverges
7. Converges
9. Converges
11. $p > 1$
13. $p > 1$
15.

Exercises 9.4

1. $\displaystyle\sum_{n=0}^{\infty} b_n$ converges; we cannot conclude anything about $\displaystyle\sum_{n=0}^{\infty} c_n$
3. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, as $1/(n^2+3n-5) \leq 1/n^2$ for all $n > 1$.
5. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, as $1/n \leq \ln n/n$ for all $n \geq 3$.
7. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Since $n = \sqrt{n^2} > \sqrt{n^2-1}$, $1/n \leq 1/\sqrt{n^2-1}$ for all $n \geq 2$.
9. Converges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.
11. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.
13. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

15. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$:

$$\frac{1}{n}=\frac{n^2}{n^3}<\frac{n^2+n+1}{n^3}<\frac{n^2+n+1}{n^3-5},$$

for all $n\geq 1$.

17. Converges; compare to $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n$, as $2^n/(5^n+10)<2^n/5^n$ for all $n\geq 1$.

19. Converges by Comparison Test with $\sum\frac{1}{n^3}$

21. Diverges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Note that

$$\frac{n}{n^2-1}=\frac{n^2}{n^2-1}\cdot\frac{1}{n}>\frac{1}{n},$$

as $\frac{n^2}{n^2-1}>1$, for all $n\geq 2$.

23. Converges; compare to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, as $1/(n^2\ln n)\leq 1/n^2$ for all $n\geq 3$.

25. Converges; Integral Test

27. Diverges; the n^{th} Term Test and Direct Comparison Test can be used.

29. Converges; the Direct Comparison Test can be used with sequence $1/3^n$.

31. Diverges; the n^{th} Term Test can be used, along with the Integral Test.

33. (a) Converges; use Direct Comparison Test as $\frac{a_n}{n}<a_n$.
    (b) Converges; since original series converges, we know $\lim_{n\to\infty}a_n=0$. Thus for large n, $a_na_{n+1}<a_n$.
    (c) Converges; similar logic to part (b) so $(a_n)^2<a_n$.
    (d) May converge; certainly $na_n>a_n$ but that does not mean it does not converge.
    (e) Does not converge, using logic from (b) and n^{th} Term Test.

Exercises 9.5

1. The signs of the terms do not alternate; in the given series, some terms are negative and the others positive, but they do not necessarily alternate.

3. Many examples exist; one common example is $a_n=(-1)^n/n$.

5. (a) converges
   (b) converges (p-Series)
   (c) absolute

7. (a) diverges (limit of terms is not 0)
   (b) diverges
   (c) n/a; diverges

9. (a) converges
   (b) diverges (Limit Comparison Test with $1/n$)
   (c) conditional

11. (a) diverges (limit of terms is not 0)
    (b) diverges
    (c) n/a; diverges

13. (a) diverges (terms oscillate between ± 1)
    (b) diverges
    (c) n/a; diverges

15. (a) converges
    (b) converges (Geometric Series with $r=2/3$)
    (c) absolute

17. (a) converges
    (b) diverges (p-Series Test with $p=1/2$)
    (c) conditional

19. $S_5=-1.1906$; $S_6=-0.6767$;
$$-1.1906\leq\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}\leq -0.6767$$

21. $S_6=0.3681$; $S_7=0.3679$;
$$0.3681\leq\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\leq 0.3679$$

23. $n=5$

25. Using the theorem, we find $n=499$ guarantees the sum is within 0.001 of $\pi/4$. (Convergence is actually faster, as the sum is within ε of $\pi/24$ when $n\geq 249$.)

27. Using 5 terms, the series in 23 gives $\pi\approx 3.142013$. Using 499 terms, the series in 25 gives $\pi\approx 3.143597$. The series in 23 gives the better approximation, and requires many fewer terms.

Exercises 9.6

1. algebraic, or polynomial.
3. Integral Test, Limit Comparison Test, and Root Test
5. Converges
7. Converges
9. The Ratio Test is inconclusive; the p-Series Test states it diverges.
11. Converges
13. Converges; note the summation can be rewritten as $\displaystyle\sum_{n=1}^{\infty}\frac{2^nn!}{3^nn!}$, from which the Ratio Test can be applied.
15. Diverges
17. Converges
19. Converges
21. Diverges

23. Diverges. The Root Test is inconclusive, but the n^{th}-Term Test shows divergence. (The terms of the sequence approach e^{-2}, not 0, as $n \to \infty$.)
25. Converges
27. Converges

Exercises 9.7

1. Diverges
3. Diverges
5. Diverges
7. Absolutely converges
9. Conditionally converges
11. Diverges
13. Absolutely converges
15. Absolutely converges
17. Absolutely converges
19. Conditionally converges
21. Absolutely converges
23. Absolutely converges
25. Diverges
27. Diverges
29. Absolutely converges
31. Diverges
33. Absolutely converges
35. Diverges
37. Absolutely converges

Exercises 9.8

1. 1
3. 5
5. $1 + 2x + 4x^2 + 8x^3 + 16x^4$
7. $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$
9. (a) $R = \infty$
   (b) $(-\infty, \infty)$
11. (a) $R = 1$
    (b) $(2, 4]$
13. (a) $R = 2$
    (b) $(-2, 2)$
15. (a) $R = 1/5$
    (b) $(4/5, 6/5)$
17. (a) $R = 1$
    (b) $(-1, 1)$
19. (a) $R = \infty$
    (b) $(-\infty, \infty)$
21. (a) $R = 1$
    (b) $[-1, 1]$
23. (a) $R = 0$
    (b) $x = 0$
25. (a) $R = 1$
    (b) $[-\frac{1}{3}, \frac{5}{3})$
27. (a) $R = \infty$
    (b) $(-\infty, \infty)$
29. $\displaystyle\sum_{n=0}^{\infty} 8^n x^{n+1}$, $R = 1/8$
31. $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+3}}{3^{n+1}}$, $R = \sqrt{3}$
33. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} n^2 x^{n-1}; \quad (-1, 1)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n}{n+1} x^{n+1}; \quad (-1, 1)$
35. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}; \quad (-2, 2)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^{n+1}; \quad [-2, 2)$
37. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!};$ $(-\infty, \infty)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}; \quad (-\infty, \infty)$
39. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} n^2 x^{n-1}; \quad (-1, 1)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{n}{n+1} x^{n+1}; \quad (-1, 1)$
41. (a) $f'(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} x^{n-1}; \quad (-2, 2)$
    (b) $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{1}{(n+1)2^n} x^{n+1}; \quad [-2, 2)$

43.

(a) $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$; $(-\infty, \infty)$

(b) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$; $(-\infty, \infty)$

45. (a) $\sum_{n=0}^{\infty} x^n(-1)^n(1+n)$; $R = 1$

(b) $\sum_{n=1}^{\infty} x^{n-1}(-1)^{n-1}(1+n)n/2$ $= \sum_{n=0}^{\infty} x^n(-1)^n(2+n)(1+n)/2$

(c) $\sum_{n=1}^{\infty} x^{n+1}(-1)^{n-1}(1+n)n/2$ $= \sum_{n=0}^{\infty} x^{n+2}(-1)^n(2+n)(1+n)/2$ $= \sum_{n=2}^{\infty} x^n(-1)^n n(-1+n)/2$

47. $\ln 3 - \sum_{n=1}^{\infty} 3^{-n} x^n/n$, $R = 3$

49. $\sum_{n=0}^{\infty} \frac{2}{2n+1} x^{2n+1}$, $R = 1$

51. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}$; $R = 1$

53. $\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} x^{n+3}$; $R = 2$

Exercises 9.9

1. The Maclaurin polynomial is a special case of Taylor polynomials. Taylor polynomials are centered at a specific x-value; when that x-value is 0, it is a Maclaurin polynomial.
3. $p_2(x) = 6 + 3x - 4x^2$.
5. $p_3(x) = 1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3$
7. $p_8(x) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$
9. $p_4(x) = \frac{2x^4}{3} + \frac{4x^3}{3} + 2x^2 + 2x + 1$
11. $p_4(x) = x^4 - x^3 + x^2 - x + 1$
13. $p_4(x) = 1 + \frac{1}{2}(-1+x) - \frac{1}{8}(-1+x)^2 + \frac{1}{16}(-1+x)^3 - \frac{5}{128}(-1+x)^4$
15. $p_6(x) = \frac{1}{\sqrt{2}} - \frac{-\frac{\pi}{4}+x}{\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^2}{2\sqrt{2}} + \frac{\left(-\frac{\pi}{4}+x\right)^3}{6\sqrt{2}} + \frac{\left(-\frac{\pi}{4}+x\right)^4}{24\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^5}{120\sqrt{2}} - \frac{\left(-\frac{\pi}{4}+x\right)^6}{720\sqrt{2}}$
17. $p_5(x) = \frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5$
19. $p_3(x) = \frac{1}{2} + \frac{1+x}{2} + \frac{1}{4}(1+x)^2$
21. $p_3(x) = x - \frac{x^3}{6}$; $p_3(0.1) = 0.09983$. Error is bounded by $\frac{1}{4!} \cdot 0.1^4 \approx 0.000004167$.
23. $p_2(x) = 3 + \frac{1}{6}(-9+x) - \frac{1}{216}(-9+x)^2$; $p_2(10) = 3.16204$. The third derivative of $f(x) = \sqrt{x}$ is bounded on $[9, 10]$ by 0.0015. Error is bounded by $\frac{0.0015}{3!} \cdot 1^3 = 0.0003$.
25. The n^{th} derivative of $f(x) = e^x$ is bounded by e on $[0, 1]$. Thus $|R_n(1)| \leq \frac{e}{(n+1)!}1^{(n+1)}$. When $n = 7$, this is less than 0.0001.
27. The n^{th} derivative of $f(x) = \cos x$ is bounded by 1 on intervals containing 0 and $\pi/3$. Thus $|R_n(\pi/3)| \leq \frac{1}{(n+1)!}(\pi/3)^{(n+1)}$. When $n = 7$, this is less than 0.0001. Since the Maclaurin polynomial of $\cos x$ only uses even powers, we can actually just use $n = 6$.
29. $\frac{1}{n!}x^n$
31. When n even, 0; when n is odd, $\frac{(-1)^{(n-1)/2}}{n!}x^n$.
33. $(-1)^n x^n$
35. $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

Exercises 9.10

1. A Taylor polynomial is a **polynomial**, containing a finite number of terms. A Taylor series is a **series**, the summation of an infinite number of terms.
3. All derivatives of e^x are e^x which evaluate to 1 at $x = 0$. The Taylor series starts $1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$; the Taylor series is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
5. The n^{th} derivative of $1/(1-x)$ is $f^{(n)}(x) = (n)!/(1-x)^{n+1}$, which evaluates to $n!$ at $x = 0$. The Taylor series starts $1 + x + x^2 + x^3 + \cdots$; the Taylor series is $\sum_{n=0}^{\infty} x^n$
7. The Taylor series starts $0 - (x - \pi/2) + 0x^2 + \frac{1}{6}(x - \pi/2)^3 + 0x^4 - \frac{1}{120}(x - \pi/2)^5$; the Taylor series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x - \pi/2)^{2n+1}}{(2n+1)!}$
9. $f^{(n)}(x) = (-1)^n e^{-x}$; at $x = 0$, $f^{(n)}(0) = -1$ when n is odd and $f^{(n)}(0) = 1$ when n is even. The Taylor series starts $1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots$; the Taylor series is $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$.
11. $f^{(n)}(x) = (-1)^{n+1} \frac{n!}{(x+1)^{n+1}}$; at $x = 1$, $f^{(n)}(1) = (-1)^{n+1} \frac{n!}{2^{n+1}}$ The Taylor series starts $\frac{1}{2} + \frac{1}{4}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \cdots$; the Taylor series is $\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{2^{n+1}}$.

13. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$

where z is between 0 and x.
If $x > 0$, then $z < x$ and $f^{(n+1)}(z) = e^z < e^x$. If $x < 0$, then $x < z < 0$ and $f^{(n+1)}(z) = e^z < 1$. So given a fixed x value, let $M = \max\{e^x, 1\}$; $f^{(n)}(z) < M$. This allows us to state

$$|R_n(x)| \leq \frac{M}{(n+1)!}\left|x^{(n+1)}\right|.$$

For any x, $\lim\limits_{n\to\infty} \dfrac{M}{(n+1)!}\left|x^{(n+1)}\right| = 0$. Thus by the Squeeze Theorem, we conclude that $\lim\limits_{n\to\infty} R_n(x) = 0$ for all x, and hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x.$$

15. Given a value x, the magnitude of the error term $R_n(x)$ is bounded by

$$|R_n(x)| \leq \frac{\max\left|f^{(n+1)}(z)\right|}{(n+1)!}\left|x^{(n+1)}\right|,$$

where z is between 0 and x. Since $\left|f^{(n+1)}(z)\right| = \frac{n!}{(z+1)^{n+1}}$,

$$|R_n(x)| \leq \frac{1}{n+1}\left(\frac{|x|}{\min z + 1}\right)^{n+1}.$$

If $0 < x < 1$, then $0 < z < x$ and $f^{(n+1)}(z) = \frac{n!}{(z+1)^{n+1}} < n!$. Thus

$$|R_n(x)| \leq \frac{n!}{(n+1)!}\left|x^{(n+1)}\right| = \frac{x^{n+1}}{n+1}.$$

For a fixed $x < 1$,

$$\lim_{n\to\infty} \frac{x^{n+1}}{n+1} = 0.$$

17. Given $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$,
$\cos(-x) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(-x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = \cos x$, as all powers in the series are even.

19. Given $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$,
$\dfrac{d}{dx}\big(\sin x\big) = \dfrac{d}{dx}\left(\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}\right) =$
$\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(2n+1)x^{2n}}{(2n+1)!} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = \cos x$. (The summation still starts at $n = 0$ as there was no constant term in the expansion of $\sin x$).

21. $1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \dfrac{x^3}{16} - \dfrac{5x^4}{128}$

23. $1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{5x^3}{81} - \dfrac{10x^4}{243}$

25. $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x^2)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{4n}}{(2n)!}$.

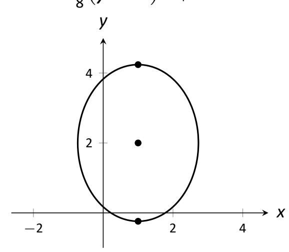
27. $\sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(2x+3)^{2n+1}}{(2n+1)!}$.

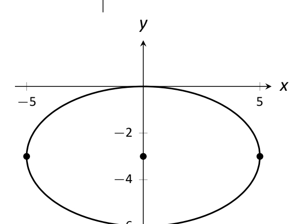
29. $x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30}$

31. $\displaystyle\int_0^{\sqrt{\pi}} \sin\left(x^2\right)\,dx \approx$
$\displaystyle\int_0^{\sqrt{\pi}} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}\right) dx = 0.8877$

Chapter 10

Exercises 10.0

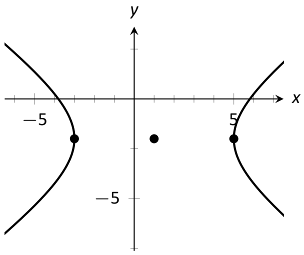
1. $y = \frac{1}{2}(x-3)^2 + \frac{3}{2}$
2. $y = \frac{-1}{12}(x+1)^2 - 1$
3. $x = -\frac{1}{4}(y-5)^2 + 2$
4. $x = y^2$
5. $y = -\frac{1}{4}(x-1)^2 + 2$
6. $x = -\frac{1}{12}y^2$
7. $y = 4x^2$
8. $x = -\frac{1}{8}(y-3)^2 + 2$
9. 

10. 

11. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$
12. $\frac{(x-1)^2}{1/4} + \frac{y^2}{9} = 1$
13. $\frac{(x-1)^2}{2} + (y-2)^2 = 1$
14. $\frac{x^2}{3} + \frac{y^2}{5} = 1$
15. $\frac{x^2}{4} + \frac{(y-3)^2}{6} = 1$
16. $\frac{(x-2)^2}{4} + \frac{(y-2)^2}{4} = 1$
17. $x^2 - \frac{y^2}{3} = 1$
18. $y^2 - \frac{x^2}{24} = 1$
19. $\frac{(y-3)^2}{4} - \frac{(x-1)^2}{9} = 1$

20. $\frac{(x-1)^2}{9} - \frac{(y-3)^2}{4} = 1$

21.



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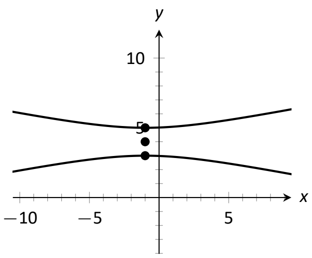


23. $\frac{x^2}{4} - \frac{y^2}{3} = 1$

24. $\frac{x^2}{3} - \frac{(y-1)^2}{9} = 1$

25. $(y-2)^2 - \frac{x^2}{10} = 1$

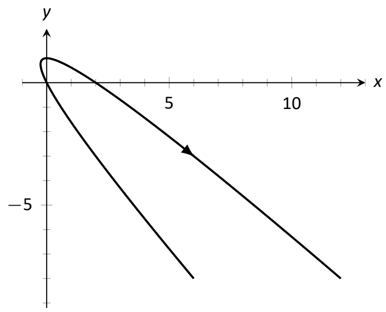
26. $4y^2 - \frac{x^2}{4} = 1$

Exercises 10.1

1. T
3. $\sqrt{2}$
5. $4/3$
7. $109/2$
9. $12/5$
11. $-\ln(2-\sqrt{3}) \approx 1.31696$
13. $\int_0^1 \sqrt{1+4x^2}\, dx$
15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\, dx$
17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\, dx$
19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\, dx$
21. 1.4790
23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y = x^2$; why?
25. Simpson's Rule fails.
27. 1.4058
29. $2\pi \int_0^1 2x\sqrt{5}\, dx = 2\pi\sqrt{5}$
31. $2\pi \int_0^1 \sqrt{x}\sqrt{1+1/(4x)}\, dx = \pi/6(5\sqrt{5}-1)$

Exercises 10.2

1. T
3. rectangular
5.



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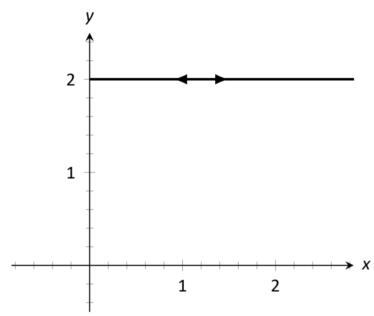

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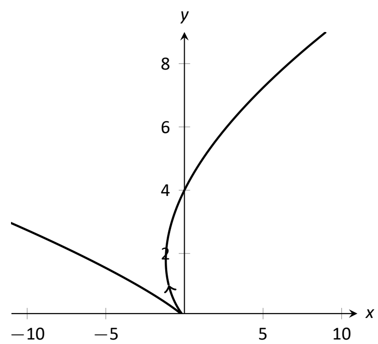

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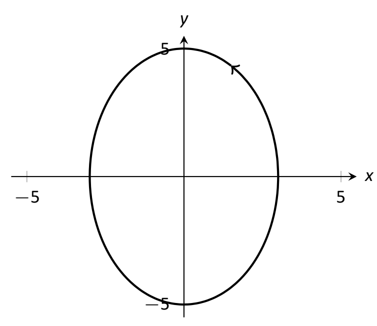

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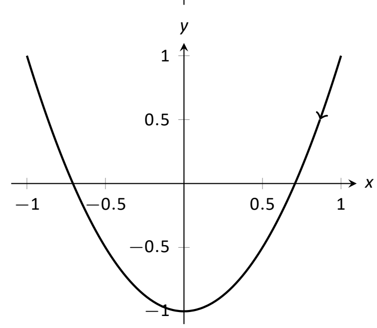
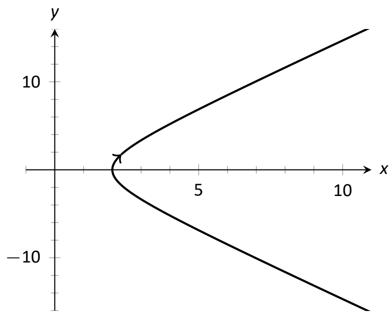
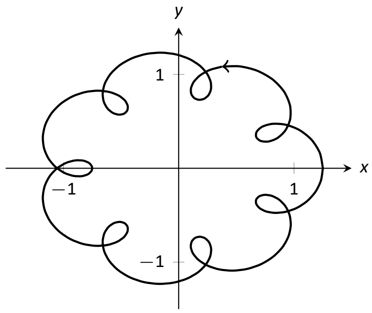


15. 

17. 

19. (a) Traces the parabola $y = x^2$, moves from left to right.
    (b) Traces the parabola $y = x^2$, but only from $-1 \leq x \leq 1$; traces this portion back and forth infinitely.
    (c) Traces the parabola $y = x^2$, but only for $0 < x$. Moves left to right.
    (d) Traces the parabola $y = x^2$, moves from right to left.

21. Possible Answer: $x = t, y = 9 - 4t$

23. Possible Answer: $x = -9 + 7\cos t, y = 4 + 7\sin t$

25. Possible Answer: $x = \frac{5}{4}t + \frac{11}{4}, y = t, [-3, 1]$

27. Possible Answer: $x = t, y = t^2 + 2t, (-\infty, -1]$

29. $x = (t+11)/6, y = (t^2 - 97)/12$. At $t = 1, x = 2, y = -8$.
    $y' = 6x - 11$; when $x = 2, y' = 1$.

31. $x = \cos^{-1} t, y = \sqrt{1 - t^2}$. At $t = 1, x = 0, y = 0$.
    $y' = \cos x$; when $x = 0, y' = 1$.

33. Possible answers:
    (a) $x = \sin t, y = \cos t, [\pi/2, 5\pi/2]$
    (b) $x = \cos t, y = \sin t, [0, 2\pi]$
    (c) $x = \sin t, y = \cos t, [\pi/2, 9\pi/2]$
    (d) $x = \cos t, y = \sin t, [0, 4\pi]$

35. $x = 4t, y = -16t^2 + 64t$

37. $x = 10t, y = -16t^2 + 320t$

39. $x = 3\cos(2\pi t) + 1, y = 3\sin(2\pi t) + 1$; other answers possible

41. $x = 5\cos t, y = \sqrt{24}\sin t$; other answers possible

43. $x = 2\tan t, y = \pm 6\sec t$; other answers possible

45. $y = -1.5x + 8.5$

47. $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$

49. $y = 2x + 3$

51. $y = e^{2x} - 1$

53. $x^2 - y^2 = 1$

55. $y = \frac{b}{a}(x - x_0) + y_0$; line through (x_0, y_0) with slope b/a.

57. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; ellipse centered at (h, k) with horizontal axis of length $2a$ and vertical axis of length $2b$.

59. $t = \pm 1$

61. $t = \pi/2, 3\pi/2$

63. $t = -1$

65. $t = k\pi$ for integer values of k

Exercises 10.3

1. F

3. F

5. (a) $\frac{dy}{dx} = 2t$
   (b) Tangent line: $y = 2(x-1) + 1$; normal line: $y = -1/2(x-1) + 1$

7. (a) $\frac{dy}{dx} = \frac{2t+1}{2t-1}$
   (b) Tangent line: $y = 3x+2$; normal line: $y = -1/3x+2$

9. (a) $\frac{dy}{dx} = \csc t$
   (b) $t = \pi/4$: Tangent line: $y = \sqrt{2}(x - \sqrt{2}) + 1$; normal line: $y = -1/\sqrt{2}(x - \sqrt{2}) + 1$

11. (a) $\frac{dy}{dx} = \frac{\cos t \sin(2t) + 2\sin t\cos(2t)}{-\sin t\sin(2t) + 2\cos t\cos(2t)}$
    (b) Tangent line: $y = x - \sqrt{2}$; normal line: $y = -x$

13. horizontal: $t = 0$; vertical: none

15. horizontal: $t = -1/2$; vertical: $t = 1/2$

17. horizontal: none; vertical: $t = 0$

19. The solution is non-trivial; use identities $\sin(2t) = 2\sin t\cos t$ and $\cos(2t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t$ to rewrite $dy/dt = 2\sin t(2\cos^2 t - \sin^2 t)$ and $dx/dt = 2\cos t(1 - 3\sin^2 t)$. Horizontal: $\sin t = 0$ when $t = 0, \pi, 2\pi$, and $2\cos^2 t - \sin^2 t = 0$ when $t = \tan^{-1}(\sqrt{2}),\ \pi \pm \tan^{-1}(\sqrt{2}),\ 2\pi - \tan^{-1}(\sqrt{2})$. Vertical: $\cos t = 0$ when $t = \pi/2, 3\pi/2$, and $1 - 3\sin^2 t = 0$ when $t = \sin^{-1}(1/\sqrt{3}), \pi - \sin^{-1}(1/\sqrt{3})$.

21. $t_0 = 0$; $\lim_{t\to 0} \frac{dy}{dx} = 0$.

23. $t_0 = 1$; $\lim_{t\to 1} \frac{dy}{dx} = \infty$.

25. $\frac{d^2y}{dx^2} = 2$; always concave up

27. $\frac{d^2y}{dx^2} = -\frac{4}{(2t-1)^3}$; concave up on $(-\infty, 1/2)$; concave down on $(1/2, \infty)$.

29. $\frac{d^2y}{dx^2} = -\cot^3 t$; concave up on $(-\pi/2, 0)$; concave down on $(0, \pi/2)$.

31. $\frac{d^2y}{dx^2} = \frac{4(13+3\cos(4t))}{(\cos t + 3\cos(3t))^3}$, obtained with a computer algebra system; concave up on $\left(-\tan^{-1}(\frac{1}{\sqrt{2}}), \tan^{-1}(\frac{1}{\sqrt{2}})\right)$, concave down on $\left(-\frac{\pi}{2}, -\tan^{-1}(\frac{1}{\sqrt{2}})\right)$; $\left(\tan^{-1}(\frac{1}{\sqrt{2}}), \frac{\pi}{2}\right)$

33. $L = 6\pi$

35. $L = 2\sqrt{34}$

37. 2π

39. $-\frac{\sqrt{10}}{3} + \ln(3+\sqrt{10}) + \sqrt{2} - \ln(1+\sqrt{2})$

41. $L \approx 2.4416$ (actual value: $L = 2.42211$)

43. $L \approx 4.19216$ (actual value: $L = 4.18308$)

45. The answer is 16π for both (of course), but the integrals are different.

47. $\frac{6\pi a^2}{5}$

49. $SA \approx 8.50101$ (actual value $SA = 8.02851$

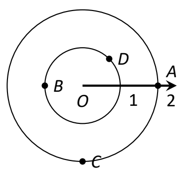
51. $\frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\theta$

Exercises 10.4

1. Answers will vary.

3. T

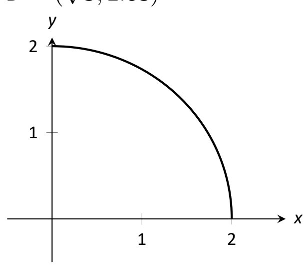
5.



7. $A(2.5, \pi/4)$ and $A(-2.5, 5\pi/4)$;
$B(-1, 5\pi/6)$ and $B(1, 11\pi/6)$;
$C(3, 4\pi/3)$ and $C(-3, \pi/3)$;
$D(1.5, 2\pi/3)$ and $D(-1.5, 5\pi/3)$

9. $A = (\sqrt{2}, \sqrt{2})$;
$B = (\sqrt{2}, -\sqrt{2})$;
$C = (\sqrt{5}, -0.46)$;
$D = (\sqrt{5}, 2.68)$

11.



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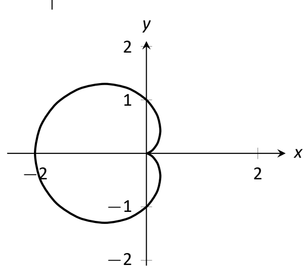

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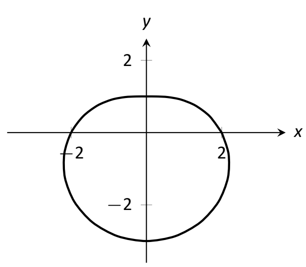

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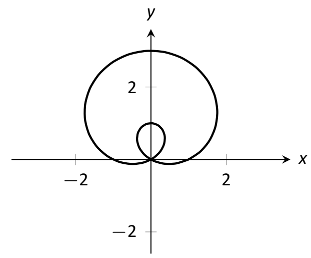
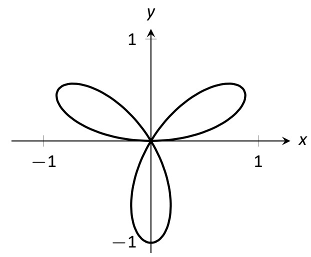


19.



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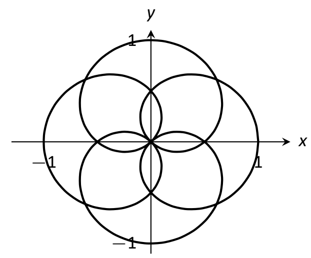

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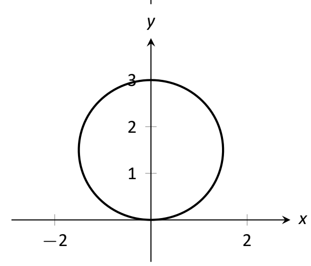

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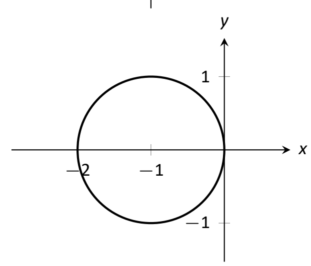

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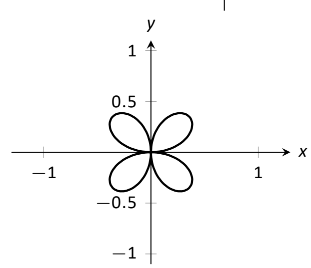

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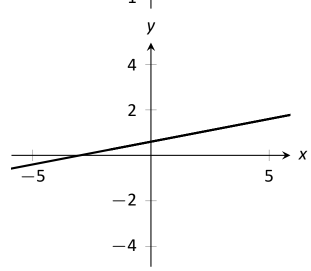
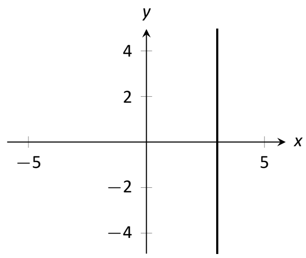


31. 

33. $(x-1)^2+y^2=1$

35. $x^2+(y-\frac{3}{2})^2=\frac{9}{4}$

37. $(x-1/2)^2+(y-1/2)^2=1/2$

39. $x=3$

41. $x^4+x^2y^2-y^2=0$

43. $x^2+y^2=4$

45. $\theta=\pi/4$

47. $r=5\sec\theta$

49. $r=\cos\theta/\sin^2\theta$

51. $r=\sqrt{7}$

53. $P(\sqrt{3}/2,\pi/6)$, $P(0,\pi/2)$, $P(-\sqrt{3}/2,5\pi/6)$

55. $P(0,0)=P(0,\pi/2)$, $P(\sqrt{2},\pi/4)$

57. $P(\sqrt{2}/2,\pi/12)$, $P(-\sqrt{2}/2,5\pi/12)$, $P(\sqrt{2}/2,3\pi/4)$, and the origin.

59. For all points, $r=1$;
$\theta=\frac{\pi}{12},\ \frac{5\pi}{12},\ \frac{7\pi}{12},\ \frac{11\pi}{12},\ \frac{13\pi}{12},\ \frac{17\pi}{12},\ \frac{19\pi}{12},\ \frac{23\pi}{12}$.

61. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

Exercises 10.5

1. Using $x=r\cos\theta$ and $y=r\sin\theta$, we can write $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$.

3. (a) $\frac{dy}{dx}=-\cot\theta$
   (b) tangent line: $y=-(x-\sqrt{2}/2)+\sqrt{2}/2$; normal line: $y=x$

5. (a) $\frac{dy}{dx}=\frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta-\sin\theta(1+\sin\theta)}$
   (b) tangent line: $x=3\sqrt{3}/4$; normal line: $y=3/4$

7. (a) $\frac{dy}{dx}=\frac{\theta\cos\theta+\sin\theta}{\cos\theta-\theta\sin\theta}$
   (b) tangent line: $y=-(2/\pi)x+\pi/2$; normal line: $y=(\pi/2)x+\pi/2$

9. (a) $\frac{dy}{dx}=\frac{4\sin(\theta)\cos(4\theta)+\sin(4\theta)\cos(\theta)}{4\cos(\theta)\cos(4\theta)-\sin(\theta)\sin(4\theta)}$
   (b) tangent line: $y=5\sqrt{3}(x+\sqrt{3}/4)-3/4$; normal line: $y=-1/5\sqrt{3}(x+\sqrt{3}/4)-3/4$

11. horizontal: $\theta=\pi/2,3\pi/2$;
vertical: $\theta=0,\pi,2\pi$

13. horizontal: $\theta=\tan^{-1}(1/\sqrt{5}),\ \pi/2,\ \pi-\tan^{-1}(1/\sqrt{5}),\ \pi+\tan^{-1}(1/\sqrt{5}),\ 3\pi/2,\ 2\pi-\tan^{-1}(1/\sqrt{5})$;
vertical: $\theta=0,\ \tan^{-1}(\sqrt{5}),\ \pi-\tan^{-1}(\sqrt{5}),\ \pi,\ \pi+\tan^{-1}(\sqrt{5}),\ 2\pi-\tan^{-1}(\sqrt{5})$

15. In polar: $\theta=0\ \cong\ \theta=\pi$
In rectangular: $y=0$

17. In polar: $\theta=\frac{\pi}{4}$ and $\theta=-\frac{\pi}{4}$
In rectangular: $y=x$ and $y=-x$.

19. area = $\frac{4\pi}{3}+2\sqrt{3}$

21. area = $\pi/12$

23. area = $3\pi/2$

25. area = $2\pi+3\sqrt{3}/2$

27. area = 1

29. area = $\frac{1}{32}(4\pi-3\sqrt{3})$

31. $x'(\theta)=f'(\theta)\cos\theta-f(\theta)\sin\theta$,
$y'(\theta)=f'(\theta)\sin\theta+f(\theta)\cos\theta$. Square each and add; applying the Pythagorean Theorem twice achieves the result.

33. 4π

35. $L\approx2.2592$; (actual value $L=2.22748$)

37. $SA=16\pi$

39. $SA=32\pi/5$

41. $SA=36\pi$

Index

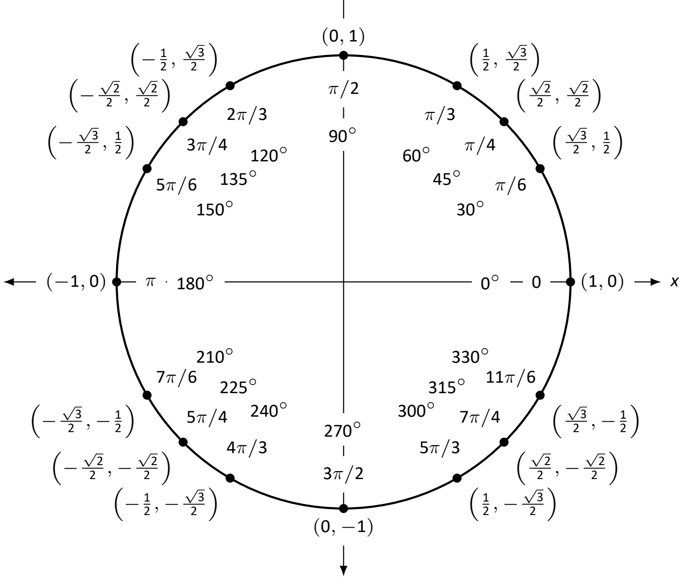
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$
7. $\frac{d}{dx}(c) = 0$
8. $\frac{d}{dx}(x) = 1$
9. $\frac{d}{dx}(x^n) = nx^{n-1}$
10. $\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}f'(x)$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x)$
13. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
14. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
15. $\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)}f'(x)$
16. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
17. $\frac{d}{dx}(\sin x) = \cos x$
18. $\frac{d}{dx}(\cos x) = -\sin x$
19. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
20. $\frac{d}{dx}(\sec x) = \sec x \tan x$
21. $\frac{d}{dx}(\tan x) = \sec^2 x$
22. $\frac{d}{dx}(\cot x) = -\csc^2 x$
23. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
24. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
25. $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
26. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
27. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
28. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
29. $\frac{d}{dx}(\cosh x) = \sinh x$
30. $\frac{d}{dx}(\sinh x) = \cosh x$
31. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
32. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
33. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
34. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
35. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
36. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
37. $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
38. $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
39. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
40. $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
2. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$
4. $\int f(g(x))g'(x)\,dx = \int f(u)\,du; \quad u = g(x)$
5. $\int 0\,dx = C$
6. $\int 1\,dx = x + C$
7. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C; \quad n \neq -1$
8. $\int e^x\,dx = e^x + C$
9. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
10. $\int \ln x\,dx = x\ln x - x + C$
11. $\int \frac{1}{x}\,dx = \ln|x| + C$
12. $\int \cos x\,dx = \sin x + C$
13. $\int \sin x\,dx = -\cos x + C$
14. $\int \tan x\,dx = -\ln|\cos x| + C$
15. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
16. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
17. $\int \cot x\,dx = \ln|\sin x| + C$
18. $\int \sec^2 x\,dx = \tan x + C$
19. $\int \csc^2 x\,dx = -\cot x + C$
20. $\int \sec x \tan x\,dx = \sec x + C$
21. $\int \csc x \cot x\,dx = -\csc x + C$
22. $\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$
23. $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$
24. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
25. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{|a|} + C$
26. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{|a|}\sec^{-1}\left|\frac{x}{a}\right| + C$
27. $\int \cosh x\,dx = \sinh x + C$
28. $\int \sinh x\,dx = \cosh x + C$
29. $\int \tanh x\,dx = \ln(\cosh x) + C$
30. $\int \coth x\,dx = \ln|\sinh x| + C$
31. $\int \sec^3 x\,dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$
32. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2+a^2}\right) + C$
33. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{x^2-a^2}\right) + C; \qquad 0 < a < x$
34. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} + C = \ln\left(x + \sqrt{x^2+a^2}\right) + C; \qquad 0 < a$
35. $\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, & |x| < |a| \\ \frac{1}{a}\coth^{-1}\frac{x}{a} + C, & |a| < |x| \end{cases} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
36. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|x|}{a} + C = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2-x^2}}\right| + C; \quad 0 < |x| < a$
37. $\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|x|}{a} + C = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right| + C; \quad x \neq 0, a > 0$

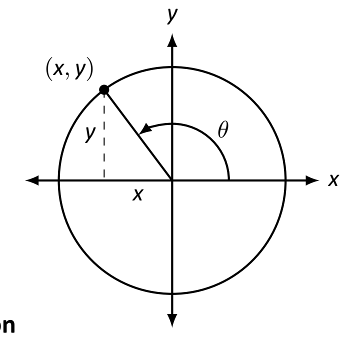
The Unit Circle



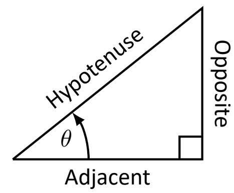
Definitions of the Trigonometric Functions

Unit Circle Definition

$$\sin\theta = y \qquad \cos\theta = x$$
$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



Right Triangle Definition



$$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$$
$$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$$
$$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin(\tfrac{\pi}{2} - x) = \cos x \qquad \csc(\tfrac{\pi}{2} - x) = \sec x$$
$$\cos(\tfrac{\pi}{2} - x) = \sin x \qquad \sec(\tfrac{\pi}{2} - x) = \csc x$$
$$\tan(\tfrac{\pi}{2} - x) = \cot x \qquad \cot(\tfrac{\pi}{2} - x) = \tan x$$

Even/Odd Identities

$$\sin(-x) = -\sin x \qquad \csc(-x) = -\csc x$$
$$\cos(-x) = \cos x \qquad \sec(-x) = \sec x$$
$$\tan(-x) = -\tan x \qquad \cot(-x) = -\cot x$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2})$$
$$\sin x - \sin y = 2\sin(\tfrac{x-y}{2})\cos(\tfrac{x+y}{2})$$
$$\cos x + \cos y = 2\cos(\tfrac{x+y}{2})\cos(\tfrac{x-y}{2})$$
$$\cos x - \cos y = 2\sin(\tfrac{x+y}{2})\sin(\tfrac{y-x}{2})$$

Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1$$
$$= 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Product to Sum Formulas

$$\sin x \sin y = \tfrac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$
$$\cos x \cos y = \tfrac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$
$$\sin x \cos y = \tfrac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$
$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

Domains and ranges of inverse trigonometric functions

Inverse Function	Domain	Range	Inverse Function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

Areas and Volumes

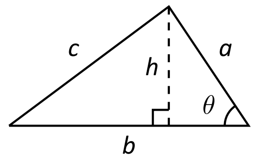
Triangles

$h = a \sin\theta$

Area $= \frac{1}{2}bh$

Law of Cosines:

$c^2 = a^2 + b^2 - 2ab\cos\theta$

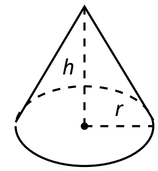


Right Circular Cone

Volume $= \frac{1}{3}\pi r^2 h$

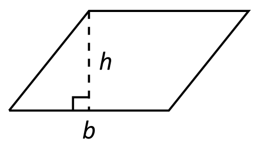
Surface Area $=$

$\pi r\sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

Area $= bh$

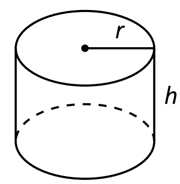


Right Circular Cylinder

Volume $= \pi r^2 h$

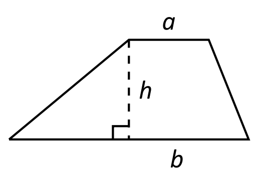
Surface Area $=$

$2\pi rh + 2\pi r^2$



Trapezoids

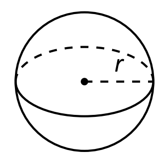
Area $= \frac{1}{2}(a + b)h$



Sphere

Volume $= \frac{4}{3}\pi r^3$

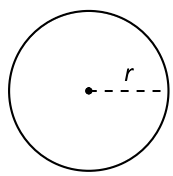
Surface Area $= 4\pi r^2$



Circles

Area $= \pi r^2$

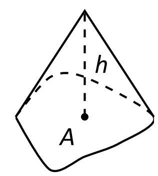
Circumference $= 2\pi r$



General Cone

Area of Base $= A$

Volume $= \frac{1}{3}Ah$

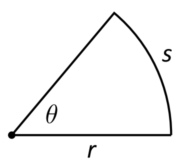


Sectors of Circles

θ in radians

Area $= \frac{1}{2}\theta r^2$

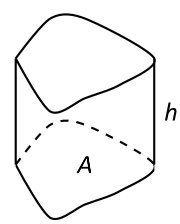
$s = r\theta$



General Right Cylinder

Area of Base $= A$

Volume $= Ah$



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, then the zeros of p are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Special Factoring

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \neq 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right]$

Simpson's Rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \cdots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

Work Done by a Variable Force

$$W = \int_a^b F(x)\,\mathrm{d}x$$

Force Exerted by a Fluid

$$F = \int_a^b w\,d(y)\,\ell(y)\,\mathrm{d}y$$

Taylor Series Expansion for $f(x)$

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Standard Form of Conic Sections

Parabola: Vertical axis	Parabola: Horizontal axis	Ellipse	Hyperbola: Foci and vertices on x-axis	Hyperbola: Foci and vertices on y-axis
$y = \dfrac{x^2}{4p}$	$x = \dfrac{y^2}{4p}$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Summary of Tests for Series

Notation: Infinite series $\sum_{n=1}^{\infty} a_n$ with sequence of partial sums $\{S_n\} = \{a_1 + a_2 + a_3 + \cdots + a_n\}$

Test	Series	Convergence or Divergence	Comment
Definition of Series	$\sum_{n=1}^{\infty} a_n$	series converges if and only if $\{S_n\}$ converges	used when a formula for S_n can be found
Divergence Test	$\sum_{n=1}^{\infty} a_n$	diverges if $\lim_{n\to\infty} a_n \neq 0$	no conclusion if $\lim_{n\to\infty} a_n = 0$
Alternating Series	$\pm\sum_{n=1}^{\infty}(-1)^n b_n$	converges if $b_n > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$	check that conditions hold eventually; no information about divergence
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	converges if and only if $\lvert r\rvert < 1$	$\text{Sum} = \frac{a}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} b_n - b_{n+m}$	converges if and only if $\{S_n\}$ converges	most terms of S_n subtract away
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	converges if and only if $p > 1$	assumes $an + b \neq 0$
p-Series For Logarithms	$\sum_{n=1}^{\infty} \frac{1}{(an+b)(\log n)^p}$	converges if and only if $p > 1$	logarithm's base doesn't affect convergence.
Integral Test	$\sum_{n=1}^{\infty} a_n$	converges if and only if $\int_k^{\infty} a(n)\,dn$ converges	$a_n = a(n)$ must be positive and decreasing eventually
Direct Comparison	$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ $0 < a_n \leq b_n$	$\sum b_n$ converges $\Rightarrow \sum a_n$ converges $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges	consider geometric or p-series
Limit Comparison	$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ $0 < a_n, b_n$	if $\lim_{n\to\infty} a_n/b_n = L$ $L > 0$: both converge or diverge together $L = 0$: $\sum b_n$ converges $\Rightarrow \sum a_n$ converges $L = \infty$: $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges	consider geometric or p-series
Ratio/Root Test	$\sum_{n=1}^{\infty} a_n$	$L = \begin{cases} \lim_{n\to\infty} \lvert a_{n+1}/a_n\rvert & \text{Ratio Test} \\ \lim_{n\to\infty} \lvert a_n\rvert^{1/n} & \text{Root Test} \end{cases}$ $L < 1$: converges $L > 1$ or $L = \infty$: diverges $L = 1$: test indeterminate	use Ratio Test for products, factorials, or powers in terms use Root Test for series of the form $a_n = (b_n)^n$

Absolute convergence: $\sum_{n=1}^{\infty} \lvert a_n\rvert$ converges (and by *Absolute Convergence Theorem*, $\sum_{n=1}^{\infty} a_n$ converges)

Conditional convergence: $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \lvert a_n\rvert$ diverges