Intersections of Polar Curves

The purpose of this supplement is to find a method for determining where graphs of polar equations intersect each other. Let’s start with a fairly straightforward example.

Example 1. The graphs of \( r = 1 \) and \( r = 2 \cos \theta \) are sketched in the figure below. Where do they intersect each other?

It appears that they intersect twice, once in the first quadrant and once in the fourth. One way that we might try and determine the intersection is by solving the following system of equations for \( r \) and \( \theta \).

\[
\begin{align*}
  r &= 1 \\
  r &= 2 \cos \theta
\end{align*}
\]

We first set these equal to each other and solve for \( \theta \), as follows:

\[
\begin{align*}
  2 \cos \theta &= 1 \\
  \cos \theta &= 1/2 \\
  \theta &= \pi/3, \ 5\pi/3
\end{align*}
\]

We only list possible solutions between 0 and \( 2\pi \) because by the time \( \theta \) has run from 0 to \( 2\pi \) each graph has been traversed at least once.
Now we use the equation \( r = 1 \) to find the \( r \)-coordinates of the points we are interested in. The intersections must occur at the points \((1, \pi/3)\) and \((1, 5\pi/3)\).

Essentially what we did in the previous example was to solve the equations simultaneously (in other words, we thought of them as a system of equations and solved that system). This is exactly what we do with rectangular equations, so there doesn’t really seem to be anything new here. Unfortunately, this method doesn’t always work. Let’s look at another example.

Example 2. Find the intersection of the graphs of \( r = \cos \theta \) and \( r = 1 - \cos \theta \). The graphs are sketched below.

There appear to be three intersection points to look for.

Let’s first try to do what we did in the previous example. Our system of equations is:

\[
\begin{align*}
r &= \cos \theta \\
r &= 1 - \cos \theta
\end{align*}
\]

These are already solved for \( r \), so we set these expressions equal to each other and solve:

\[
\begin{align*}
\cos \theta &= 1 - \cos \theta \\
2 \cos \theta &= 1 \\
\cos \theta &= 1/2 \\
\theta &= \pi/3, \ 5\pi/3
\end{align*}
\]

Now plug these values for \( \theta \) into the first equation to find the intersection points \((1/2, \pi/3)\) and \((1/2, 5\pi/3)\).

Is there really a third point of intersection? Looking at the graph, it seems that we are missing an intersection point at the origin. We can check to see
whether or not both curves really go through the origin by considering the curves separately. The curve \( r = \cos \theta \) passes through the origin when \( r = 0 \) and \( \theta = \pi/2 \). The curve \( r = 1 - \cos \theta \) passes through the origin when \( r = 0 \) and \( \theta = 0 \). Since both curves pass through the origin, this is another point of intersection.

The points of intersection are \((0,0)\), \((1/2, \pi/2)\), and \((1/2, 5\pi/3)\).

In order to understand what sometimes makes polar equations more difficult to deal with than rectangular equations, we must understand what happened in the previous example. When we determined whether or not the origin was on both curves, we found that \((0, \pi/2)\) was on one curve and \((0,0)\) was on the other. These ordered pairs are different because they have different second coordinates, but they both represent the same point. The reason that polar equations can be difficult to work with is that the same point can be represented by more than one ordered pair of polar coordinates.

When we solve a system of equations in two unknowns, we find ordered pairs that satisfy all of the equations. If the equations are rectangular equations for curves, then this is equivalent to finding the points of intersection of the curves because each point in the plane is associated with only one ordered pair of rectangular coordinates. If the equations are polar equations of curves, then we only find the points of intersection that happen to be represented by the same ordered pair in both equations. We miss any intersection points that are represented by different ordered pairs in different equations.

How do we solve this problem? Unfortunately, there isn’t a single method that always works best. One of the things that we might do is what we did in the last example: carefully sketch the graphs of the curves and use the graph to help locate intersection points. In this case we must make sure that we use the equations to verify suspected intersections. One warning about this method: if your graphs are inaccurate, you may either miss intersections or spend hours trying to locate and verify intersections that don’t exist.

The other method that will sometimes work is to translate the equations into rectangular coordinates, solve the system of translated equations, then translate the intersection points back into polar coordinates. Since there is no ambiguity with rectangular coordinates, this method will yield all of the points of intersection. For many polar curves the resulting system of translated equations will be prohibitively difficult to solve, but when you can solve the translated system this method works very well.

Let’s look at another example.

**Example 3.** Find the points of intersection of the circle \( r = 2 \cos \theta \) and the cardioid \( r = 1 + \cos \theta \). The graphs are sketched below for reference.

If you solve the system of polar equations (you can try this), you find the intersection point \((2,0)\). Let’s try converting the equations into rectangular coordinates and then solving.
First the circle:

\[ r = 2 \cos \theta \]
\[ r^2 = 2r \cos \theta \]
\[ x^2 + y^2 = 2x \]

Now the cardioid:

\[ r = 1 + \cos \theta \]
\[ r^2 = r + r \cos \theta \]
\[ x^2 + y^2 = \sqrt{x^2 + y^2 + x} \]

Our system of equations in rectangular coordinates is:

\[ x^2 + y^2 = 2x \]
\[ x^2 + y^2 = \sqrt{x^2 + y^2 + x} \]

To solve this, we use the first equation to replace all occurrences of \( x^2 + y^2 \) in the second equation with \( 2x \), then solve:

\[ 2x = \sqrt{2x + x} \]
\[ x = \sqrt{2x} \]
\[ x^2 = 2x \]
\[ x^2 - 2x = 0 \]
\[ x(x - 2) = 0 \]
\[ x = 0, \ 2 \]
Now plug these values back into the first equation to find the $y$-coordinates of the intersection points:

\begin{align*}
0^2 + y^2 &= 2(0) \\
0 + y^2 &= 0 \\
y^2 &= 0 \\
y &= 0
\end{align*}

and

\begin{align*}
2^2 + y^2 &= 2(2) \\
4 + y^2 &= 4 \\
y^2 &= 0 \\
y &= 0
\end{align*}

The rectangular coordinates of the points of intersection are $(0,0)$ and $(2,0)$. Translating back into polar coordinates we find the intersections of the original curves are $(0,0)$ and $(2,0)$.

Before going any further, note that converting back into polar coordinates is an important step. The fact that the rectangular and polar coordinates for the points of intersection in the above example were the same is accidental and will not generally happen.\footnote{In fact, this leads to an interesting question. If a point is on the $x$-axis, then it’s rectangular coordinates are also valid polar coordinates for the point (you should be able to show this). Is this true for any other points in the plane?}

To summarize, finding the points of intersection of the graphs of polar equations is made more difficult by the fact that the polar coordinates of a point are not unique. We can use an accurate graph to try to locate intersections, then use the equations to verify the points of intersection. Converting the equations into rectangular coordinates, solving the resulting system, then converting back into polar coordinates is another method that might be helpful.

**Problems:** Decompose each of the following fractions.

Find the points of intersection of the following pairs of curves.

1. $r = 2$, \hspace{1em} $r = 2 \cos \theta$
2. $r = \sin \theta$, \hspace{1em} $r = 1 + \sin \theta$
3. $r = \sin 2\theta$, \hspace{1em} $r = 2 \sin \theta$
4. $r = 2 + 2 \sin \theta$, \hspace{1em} $r = 2 - 2 \cos \theta$
5. $r = \cos 3\theta$, \hspace{1em} $r = \sin 3\theta$
6. $r = \sin 2\theta$, \hspace{1em} $r = \sin 3\theta$
7. \( r = 3 \cos \theta \), \( r = 2 - \cos \theta \)

8. \( r^2 = \sin 2\theta \), \( r^2 = \cos 2\theta \)

9. Find all of the self-intersections of the spiral \( r = \theta \) (see graph above).
Solutions

1. \((2, 0)\)

3. \((0, 0)\)

5. \((0, 0), \left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right), \left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right), \left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right)\)

7. \((\frac{3}{2}, \frac{\pi}{3}), \left(3, \frac{5\pi}{3}\right)\)

9. The self-intersections occur at all points of the form \(\left(\frac{\pi}{2}(2n + 1), \frac{\pi}{2}(2n + 1)\right)\) where \(n\) is a positive integer.