Applied Problems for Calculus I

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Objectives of Assignment

1. To use your knowledge of Calculus and your computer skills to solve realistic application problems.

2. To write formal technical reports describing your methods and solutions.

I. A Riveting Choice

Suppose you get a summer Co-op position with an aircraft manufacturer. Your team chief tells you that each airplane requires 35,760 special rivets, and a decision must be made on whether to make the rivets out of zinc or out of titanium.

Each rivet is exactly one centimeter long and they are ground out on a lathe. As the rivet is spun on the lathe, the template for grinding it to the right shape is given by the following function:

\[
f(x) = \begin{cases} 
\cos x^2 & \text{for } 0 \text{ cm } \leq x \leq \sqrt{\frac{\pi}{4}} \text{ cm}, \\
\sin x^2 & \text{for } \sqrt{\frac{\pi}{4}} \text{ cm } \leq x \leq 1 \text{ cm}.
\end{cases}
\]

Your task, according to the team chief, is to determine how many kilograms lighter each plane will be if titanium is used instead of zinc.

Submit a report which includes your answer, a detailed account of how you arrived at your answer, a precise three dimensional sketch of the rivet, and a bibliography in order to reference any facts you had to look up.
II. A Maximum Dose from Calculus

Suppose you are working for a pharmaceutical firm which has developed an experimental drug for treating pancreatic cancer. This drug slows the growth of malignant tumors and sometimes causes the tumors to shrink. Optimal dosage levels for humans have not yet been determined.

The more of this drug that’s in the bloodstream, the more effective it is in combating the cancer. Unfortunately, if the average blood level is above 0.5 milligrams per liter (mg/l) during the first five hours after receiving the dose, irreversible renal failure results. This is fatal and is not a desirable result.

Through experimental trials, it has been found that when this drug is administered intravenously, and an initial blood level of $Z$ milligrams per liter is established, the drug is metabolized at a rate such that the blood level is $y = Z \cdot e^{-t}$ mg/l after $t$ hours.

What is the maximum initial value of $Z$ that can be used without inducing renal failure? Find $Z$ to the nearest 0.01 mg/l. Explain how you arrived at your answer.
III. The Bell–Shaped Curve

The bell–shaped curve which occasionally reflects the shape of the distribution of exam scores in calculus classes is called the *Normal Distribution*. Many natural and man–made phenomena have this bell–shaped distribution. IQ’s, the life of steel–belted radial tires, the length of elephant tails, and the mean time between failures of computer components all demonstrate variations of this bell–shaped curve which is called the normal distribution.

There are two important parameters concerning this curve. These are where the maximum occurs - which is just the average, and how spread out the scores are - which is measured by the standard deviation. Intuitively, it should make sense that the high point on the curve should occur right around where the average score is. The average, or mean, is usually denoted by \( \mu \) (mu), and the standard deviation is usually denoted by \( \sigma \) (sigma).

The function which defines this curve is:

\[
f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\]

Notice that \( x \) only shows up once in the function and that all \( \mu \) does is shift the curve along the \( x \)–axis. Upon first being introduced to this function, students commonly wonder what business it is of \( \sqrt{2\pi} \) to be showing up in there. Example 1 answers this question.

**Example 1.** Graph the normal distribution with \( \mu = 0 \) and \( \sigma = 1 \), and then try to guess the area under this curve from \(-\infty\) to \(\infty\).

**Solution:** Graph \( f(x) = \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{x^2}{2}} \) in the window \((-5,5,0,1)\). Hopefully, it’s bell–shaped. If not, you typed something wrong. Next, type and enter “integ(f(x), x = -10, 10)”. This approximates the area under the curve on the interval \([-10, 10]\). The area will be in the output box back on the input screen. Now find the area on \([-100, 100]\). Then find the area on \([-1000,1000]\).

The \( \sqrt{2\pi} \) causes the area to be 1. No matter what values you use for \( \mu \) and \( \sigma \), the area will always be 1. \( \mu \) just moves the curve left and right, and different values of \( \sigma \) just flatten or sharpen the profile of the bell shape, but the area will always be 1. Magic.

The bell–shaped curve, or the normal distribution, has an important and interesting application in probability and statistics. Suppose the scores on a calculus exam have a normal
distribution with mean 70 and standard deviation 10 (µ = 70, σ = 10). If you want to find what percentage of the students got scores between 60 and 80, you would only need to find the area under this bell–shaped curve between 60 and 80. That is, the area under the normal curve for a given interval is the same as the percentage of students who scored in that range.

Similarly, if you want to know the probability of getting a score greater than 90, just find the area under the curve from 90 to ∞. Remember that the total area under these curves (from −∞ to ∞) is 1. This means 100% of the scores were somewhere in this range. No surprise, but that is why it was important for the area to be 1. However, it might be a little surprising that \( \sqrt{2\pi} \) causes it to be 1.

Example 2. A manufacturer claims their power supply units last on average 1,000 hours. The lives of the units are normally distributed with a standard deviation of 100 hours.

a) Graph this normal distribution.

b) Find the probability that one of these units will last more than 1,200 hours.

c) NASA engineers need a back–up power supply unit and want to be 99.8% sure it will last at least 750 hours. Does this manufacturer’s unit qualify?

Solution: IMPORTANT HINT – type in 
\[
\left(1 / (\sqrt{2\pi} s^2)\right) \times e^{-(x-\mu)^2 / (2 s^2)}
\]
and then put “u = 1000” on a separate line and “s = 100” on another line. This way, when you want to modify the function for later problems, you can just change the lines with \( u \) and \( s \).

a) Try window(500,1500,0,0.005). The reason for graphing this before having X(PLORE) integrate it is to provide an easy visual means of checking whether you entered the function correctly. It should be bell–shaped with the maximum (the mean) at 1000. Notice that the function doesn’t get very high - less than 0.005, but it is stretched out over several hundred units (really over the whole number line) so the area is still 1.

b) Using “integ(f(x), x = 1200, 2500)” you get 0.0227501. Using “integ(f(x), x = 1200, 3500)” you still get 0.0227501. Going from 1200 to +∞ would likely also give this value if rounded to six significant digits. The probability the unit will last more than 1,200 hours is 2.27501%.

c) Using “integ(f(x), x = –1000, 750)” gives 0.00620967. This means about 0.62% will fail in under 750 hours. Consequently, about 99.38% will last more than 750 hours. This does not satisfy NASA’s requirement of 99.8%. (Using x = –2000, 750 doesn’t change.)

Example 3. Company X claims their widgets last on average 600 days with a standard deviation of 25 days. Company Z claims their widgets last on average 500 days with a standard deviation of 75 days. Which company’s widgets have a better chance of lasting more than 675 days? Assume both distributions are normal.
Solution: First set $u = 600$ and $s = 25$. Using “integ(f(x), x = 675, 3000)” you get $0.0013499$. This means company X’s widgets have a 0.13499% of lasting over 675 days. Next set $u = 500$ and $s = 75$. Using the same “integ” gives $0.00981$, so company Y’s widgets have a 0.981% probability of lasting more than 675 days. Company Y wins.

(You should check $x = 675$, 5000 or something to make sure you are adequately approximating $x = 675, \infty$.)

Your task:

The IQ of humans is normally distributed with a mean of 100 and a standard deviation of 15. A very well known society which is supposedly for people with high IQ’s limits membership to individuals with an IQ at least two standard deviations above the average. What percentage of the general population can belong to this society? What IQ is the cut–off for this society?

There is another society, less well known, which is called the 4–sigma society. This group only admits individuals with IQ’s which are four standard deviations above average. What percentage of the general population can belong to this society? If the population of North Dakota is 630,000, how many North Dakotans could be expected to belong? What IQ is the cut–off for this group?

Submit a report which gives the answers to these questions and provides an explanation of how you found each answer.
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You’ve learned enough calculus for one semester.