

**Anti-Differentiation**

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### Objectives of Assignment

1. To recognize simple patterns which occur when the chain rule is used to differentiate functions.

2. To learn to anticipate these patterns and compensate for them when anti-differentiating some simple types of functions.

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### I. Introduction

In math classes you are often expected to know material so well that you can do everything forwards and backwards. Think about it for a minute: after learning addition you had to learn to do it backwards (subtraction), after learning multiplication of polynomials you had to learn to do it backwards (factoring polynomials), after learning the trig functions you had to learn their inverses.

The same will be expected of you with differentiation. You will have to know differentiation so well that you can do it backwards. That is, instead of being given a function and then be expected to determine its derivative, you will be given a function that has already been differentiated and then be expected to determine what the original function was before it had been differentiated. Finding the original non-differentiated function is called anti-differentiation.

For example, one anti-derivative of $2x$ is $x^2$, and one anti-derivative of $\sec^2 x$ is $\tan x$. You have to think in reverse and figure out what you would have to differentiate in order to arrive at the given function. It’s really just differentiation in reverse.

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### II. Examples and the Importance of Anticipating and Compensating

Some functions are easily anti-differentiated by simply using your memory. For example, you should have no difficulty coming up with the anti-derivatives of $f(x) = \cos x$, $g(x) = -\sin x$, and $h(x) = 3x^2$. Their general anti-derivatives are $\sin x + C$, $\cos x + C$, and $x^3 + C$ respectively, where “$C$” can be any constant.
To determine the anti–derivative of slightly more complicated functions, it is helpful to notice a simple pattern resulting from the chain rule. Examine the following pairs of derivatives:

\[ a) \quad \frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} 3\sin x = 3\cos x \]

\[ b) \quad \frac{d}{dx} \tan x = \sec^2 x, \quad \frac{d}{dx} 3\tan x = 3\sec^2 x \]

\[ c) \quad \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx} \sqrt[3]{x} = 3\frac{1}{2\sqrt[3]{x}} \]

\[ d) \quad \frac{d}{dx} (x)^2 = 2x, \quad \frac{d}{dx} (3x)^2 = 3 \cdot 2(3x) \]

In each of these pairs, multiplying 3 times \( x \) inside the function changed the derivative. The Chain Rule had to be used so the 3 seems to pop out in front of the derivative. This should be old hat to you.

The importance of this pattern, where the Chain Rule pops the 3 out in front of the answer, becomes a critical idea to keep in mind when you move on to anti–differentiation. This type of occurrence must be anticipated.

For example, what is the anti–derivative of \( \cos 2x \)? It is not \( \sin 2x + C \) because

\[ \frac{d}{dx} \sin 2x + C = 2\cos 2x. \]

The Chain Rule pops the 2 out and you don't get the \( \cos 2x \) that you wanted.

You must anticipate that the 2 will pop out when you are checking your answer. Once you realize that the 2 pops out, you can compensate for it. You compensate by using a \( \frac{1}{2} \) which will cancel the 2. This means the correct anti–derivative is \( \frac{1}{2} \sin 2x + C \).

Checking this anti–derivative gives

\[ \frac{d}{dx} \frac{1}{2} \sin 2x + C = \frac{1}{2} \cdot 2\cos 2x \]

\[ = \cos 2x \]
By anticipating and compensating, many anti–derivatives can be easily determined if you use just a slight bit of foresight and cleverness. A strong dose of practice will really help, too.

III. Practice Problems

Feel free to use the diff(f(x), x) command in X(PLROE) to expedite checking your answers. If your answer isn’t quite right, try to tweak it a bit so that you get the right derivative when checking. To save time, use a separate line with “f(x) = _____” and then make minor changes to it instead of typing the new function for each problem.

1. Find the anti–derivatives of the following functions:

   a) \( f(x) = \cos x \)

   b) \( f(x) = 5 \cos 5x \)

   c) \( f(x) = \cos 5x \)

   d) \( f(x) = \cos \frac{x}{2} \)

   e) \( f(x) = \sec^2 x \cdot \cos(\tan x) \) (hint: this is really no different than part b) above - what must have popped out?)

   f) \( f(x) = \cos x \cdot \cos(\sin x) \)

   g) \( f(x) = \cos x \cdot \cos(2 \sin x) \)

   h) \( f(x) = 2x \cdot \cos(x^2) \)

   i) \( f(x) = 3x \cos(x^2) \)

   k) \( f(x) = \frac{\cos \sqrt{x}}{2\sqrt{x}} \)

   l) \( f(x) = \frac{4 \cos \sqrt{x}}{\sqrt{x}} \)

2. Find the anti–derivatives of the following functions:
a) \( f(x) = 5(x + 1)^4 \)

b) \( f(x) = (x+1)^4 \)

c) \( f(x) = 5(2x+1)^4 \)

d) \( f(x) = 2(5x+1)^4 \)

e) \( f(x) = 10x \cdot (x^2+1)^4 \)

f) \( f(x) = x \cdot (2x^2+1)^4 \)

g) \( f(x) = \frac{5(\sqrt{x} + 1)^4}{2\sqrt{x}} \)

h) \( f(x) = \frac{(\sqrt{x} + 1)^4}{\sqrt{x}} \)

3. Find the anti–derivatives of the following functions:

a) \( f(x) = \frac{1}{2\sqrt{x}+1} \)

b) \( f(x) = \frac{1}{\sqrt{2x+1}} \)

c) \( f(x) = \frac{x}{\sqrt{x^2+1}} \)

d) \( f(x) = \sec^2 4x \)

e) \( f(x) = 3x^2 \sec^2 x^3 \)

f) \( f(x) = x^2 \sec^2 x^3 \)

g) \( f(x) = 2x^2 \sin x^3 \)