Objectives of Assignment

1. To provide a concrete basis upon which to build a solid understanding of the "delta–epsilon" definition of a limit.
2. To graphically explore limits of functions.

I. The Story So Far ...

Once upon a time, in a far away kingdom, an evil sorcerer named Al imprisoned the Fair Prince Ed in a high, high tower. There was but one window in the Prince's room at the top of the tower, and no stairs leading there. You see, nefarious Al had cast a magical spell over the tower so that Fair Prince Ed had but one chance to escape: if her royal highness the Duchess Delta could enter his room and shake his hand, they would both be set free.

Now, the only means available to Duchess Delta for getting through that lone window was a catapult. If the catapult was placed at the perfect distance from the tower, Delta could hop on it and safely get flung right through the middLe (the capital "L" is important) of the window. This seemed way too easy a solution for an exciting fairy tale, so Al decided to curse that perfect spot. If the catapult was placed at the perfect, but cursed spot, Delta would turn into a frog. This adds a little more conflict to the storyline.

Well, as luck would have it, the window in Ed's room was a little bit bigger than most windows. Consequently, even if Delta didn't sail through the exact middLe of the window, she could still safely fly into the room. In fact, the window was so big, that if she were within 6 feet of the middLe of the window, she was still OK. Coincidentally, Ed was 6 feet tall.

So, placing the catapult real, real close to Al's spot – but not right at Al's spot – could prevent Delta from turning into a frog and might still launch her safely within 6 feet of the middLe of the window. The moral of this story is really a question:

*How close to Al's spot does Delta have to be in order to fly within Ed's height of the middLe?*
II. Back to Real Life and Calculus for the Answer

Suppose that Figure 1 below represents Delta and Ed's predicament. Here the middle of the window is 123 feet above ground level, and Al's spot is 10 feet away from the base of the tower.

![Figure 1. Delta and Ed's Predicament](image)

Now let's call the distance from the tower to the catapult $x$. If $x = 10$, Delta is a frog because that's where Al's cursed spot is. Let's use $y$ to represent how high Delta lands on the tower. Delta and Ed want $y$ to be within 6 of 123. Finally, suppose that this catapult came with a certificate from the factory that guarantees $y = x^2 + 2x + 3$.

The question can now be stated as, "How close does Delta have to be to Al's spot in order to ensure $y$ will be within Ed (6 feet) of the middle?"

In calculus, we represent $y$ as a function of $x$ and say $y = f(x) = x^2 + 2x + 3$. We will use Greek letters to represent Delta and Ed, “$\delta$” for Delta and “$\varepsilon$” (epsilon) for Ed. Finally, just use "$a$" for Al's spot, and use "L" for middle. Now we can ask: "How close must $\delta$ be to $a$ in order to ensure $f(x)$ is within $\varepsilon$ of L?" Graphically, the above figure becomes the axis system below:

![Figure 2. A graphical representation of Delta and Ed's predicament](image)
III. X(PLORE) to the Rescue!!!

Get X(PLORE) running on your computer and then load "assign2.xpl". (Refer to Computer Assignment #1 if you don't recall how to do this.) Type in the function $f(x) = x^2 + 2x + 3$. Next type in the following: "limit(f(x), 0, 13, 0, 150, 6)". This will graph $f(x)$ using window(0, 13, 0, 150) and also do a little magic to help you solve Delta and Ed's dilemma. The '6' is the value of $\varepsilon$ (Ed's height). Press $\text{Enter}$.

The screen should display a graph of $f(x)$ along with a question asking what you want to use for '$a$'. Type "10" and press $\text{Enter}$. Al's spot should now be marked on the x-axis, with "L + $\varepsilon$" and "L - $\varepsilon$" marked on the y-axis. The computer also asks you for a value of $\delta$. Although you don't know what to use for $\delta$ yet, with this computer program you can use trial and error without having to worry about missing the window and hurling Delta headlong into the tower's brick wall. Such are the benefits of computer simulations.

Type in "0.5" as a first guess for $\delta$ and press $\text{Enter}$. Now the computer has marked 9.5 and 10.5 on the x-axis and has drawn a line from each of these points up to the graph of the function. The computer also asks if you want to see the corresponding values of $y = f(x)$ for this $\delta$. Simply press $\text{Enter}$ and horizontal lines are drawn from the graph to the y-axis. It is these horizontal lines which show where Delta will impact for $\delta = 0.5$.

Unfortunately, these lines will be outside of the window (L - $\varepsilon$ to L + $\varepsilon$) so Delta will smack into the wall if the catapult is 0.5 ft. away from $a$. Try another, smaller $\delta$. Try 0.4. If 0.4 doesn't get Delta through the window, try again and again. You should eventually find that if $\delta$ is 0.2 or smaller, then Delta will be within 6 feet of L and they will live happily ever after.

Using smaller and smaller values of $\delta$ really just means that the numbers being put into $f(x)$ are getting closer and closer to Al's spot ($a = 10$). As these values are getting closer to $a = 10$, the value of $f(x)$ gets closer and closer to the middle of the window (L = 123). That is, as $x$ approaches 10, $f(x)$ approaches 123. In this case we say that as $x$ approaches 10, the limit of $f(x)$ is 123. This is expressed mathematically as:

$$\lim_{{x \to 10}} f(x) = 123$$

Although this may seem like frivolous busywork, it really gets to the heart of the "delta-epsilon" definition of limits, and stresses that when you evaluate limits you are analyzing the function close to Al's spot, not right at $a$. Turning into a frog is impossible, but so is evaluating a function like $f(x) = \frac{x^2 - 1}{x - 1}$ at $x = 1$. So don't try doing either one.
IV. Examples

Example 1. Finding a relationship between $\delta$ and $\varepsilon$.

Let $f(x) = 3x - 1$ and $a = 2$. What is the largest value of $\delta$ which still keeps $f(x)$ within $\varepsilon$ of 5 if:

a) $\varepsilon = 0.3$,

b) $\varepsilon = 0.09$,

c) $\varepsilon = 0.06$,

d) what appears to be the relationship between $\delta$ and $\varepsilon$?

e) what would the largest possible value of $\delta$ be if $\varepsilon = \frac{\pi}{100}$?

Solution:

Type $f(x) = 3x - 1$ and press $\text{Enter}$. Enter "limit(f(x),1,3,4,6,0.3)". You will have to determine an appropriate window in the exercises.

a) Enter 2 for $a$, and then try 0.3 for $\delta$. This gives values outside the desired interval on the y-axis, so try a smaller value for $\delta$. When $\delta = 0.1$, it lines up perfectly. This means that if $x$ is within 0.1 of 2, then $f(x)$ will be within 0.3 of 5.

b) Go back to the input screen and change 0.3 to 0.09 and press $\text{Enter}$. Again enter 2 for $a$, and then try several values for $\delta$. The answer should be 0.03.

c) The answer here is 0.02.

d) Notice the pattern from a), b), and c):

- when $\varepsilon = 0.3$, $\delta = 0.1$;
- when $\varepsilon = 0.09$, $\delta = 0.03$;

- when $\varepsilon = 0.06$, $\delta = 0.02$. Apparently, $\delta = \frac{\varepsilon}{3}$.

e) Following the pattern, $\delta = \frac{\pi}{100} = \frac{\pi}{300}$.

Example 2. Using X(PLORE) to guess the limit of a function.

Graphically approximate the value of $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Solution:

The function $f(x) = \frac{x^2 - 1}{x - 1}$ is undefined at $x = 1$ since $f(1) = \frac{0}{0}$.

However, for values close to $x = 1$, $f(x)$ is defined and "behaves" nicely. Enter the function and then type in "limit(f(x),0,4,0,4,0)". Let $\varepsilon = 0$ here and be very careful how you enter the function.

Progressively smaller values of $\delta$ such as 0.5, 0.1, and 0.01 confine $f(x)$ to progressively narrower regions of the y-axis. Using the crosshairs, it appears $f(x)$ approaches 2 (a little less than 2.0200) as $x$ approaches 1.
Example 3. A limit that does not exist.

Graphically approximate the value of \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \).

Solution:

Enter the new function. First try \( \text{limit}(f(x),-\pi/4,\pi/4,-1,1,0) \), put in 0 for \( a \), and then try 0.1, 0.05, 0.01, and 0.005 for \( \delta \). As \( x \) approaches 0, \( f(x) \) doesn't appear to close in on some point on the y-axis. Go back to the input screen and change the window to \( (-0.1, 0.1, -1, 1) \). Values such as 0.05, 0.04, 0.03 . . . for \( \delta \) again seem to suggest there is no single value towards which \( f(x) \) converges as \( x \) approaches 0. When this happens, we say that the limit does not exist.

V. Practice Problems

1. Let \( f(x) = 1 - \frac{x}{2} \). At \( x = 6 \), what is the maximum value of \( \delta \) if:
   a) \( \epsilon = 0.5 \)?
   b) \( \epsilon = 0.2 \)?
   c) \( \epsilon = 0.05 \)?
   d) Is there a pattern in the relationship between \( \delta \) and \( \epsilon \)?
   e) \( \epsilon = \frac{39\pi}{500} \)

2. Let \( f(x) = -3x^2 + 2x + 4 \). At \( x = 0.5 \), what is the maximum value of \( \delta \) if:
   a) \( \epsilon = 0.5 \)?
   b) \( \epsilon = 0.2 \)?
   c) \( \epsilon = 0.05 \)?
   d) Is there an obvious pattern in the relationship between \( \delta \) and \( \epsilon \)?

3. Let \( f(x) = \frac{\sin(x)}{x} \). Use X(PLORE) to graphically attempt estimating \( \lim_{x \to 0} f(x) \).

4. Use X(PLORE) to graphically attempt estimating: \( \lim_{x \to 0} x^2 \sin(200x) \). Use \( f(x) = root(x,3)^2 \sin(200x) \). Window choice will be critical here.

5. Use X(PLORE) to graphically attempt estimating: \( \lim_{x \to 3} \frac{x-2}{x^2-9} \).

Answers: 1. a) 1.0, b) 0.4, c) 0.1, d) \( \delta = 2\epsilon \), e) \( \delta = \frac{39\pi}{250} \); 2. a) 0.27, b) 0.14, c) 0.043, d) no; 3. 1, 4. 0, 5. the limit does not exist.